Some more results on *qmethod*

Probabilistic estimation of $M_{\rm G}$, Z and $\log g$

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Abstract

qmethod is a Bayesian method for estimating stellar parameters using (spectro)photometric and astrometric information. Here I derive a few more results which broaden its range of application: estimation of the absolute magnitude; generalization to include dependence on and estimation of metallicity and surface gravity.

Document History

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TABLE 1: Notation

G	apparent magnitude in the G band (mag)
$M_{\rm G}$	absolute magnitude in the G band (mag)
$A_{\rm G}$	extinction in the G band (mag)
A_0	extinction parameter (mag)
T	stellar effective temperature (K)
Z	stellar metallicity (fraction)
g	surface gravity (ms^{-2})
R	stellar radius (m)
m	stellar mass (kg)
L	stellar luminosity (W)
ϖ	parallax (arcsec)
q	$\equiv G + 5\log\varpi$ (mag)
$\boldsymbol{p}$	normalized spectral energy distribution
P	probability density
log	base 10 logarithm

1 Introduction

The *qmethod* algorithm was introduced in CBJ-049 as a Bayesian method for estimating the stellar parameters of a star using not only the spectral energy distribution, but also the parallax, apparent magnitude and the astrophysical knowledge embodied in the Hertzprung–Russell Diagram (HRD). The method ensures that the inferred effective temperature and absolute magnitude are consistent with the distance and apparent magnitude. It also allows one to better determine the line-of-sight extinction by reducing the degeneracy between this and temperature.

In that TN I derived the following

$$P(A_0, T|\boldsymbol{p}, q) = \underbrace{P(\boldsymbol{p}|A_0, T)}_{\text{likelihood}} \underbrace{\frac{P(A_0)}{P(\boldsymbol{p}, q)}}_{\text{priors}} \underbrace{\int_{M_{\rm G}} \underbrace{P(q|M_{\rm G}, A_0, T)}_{\text{q constraint}} \underbrace{P(M_{\rm G}, T)}_{\text{HRD prior}} \, dM_{\rm G}}_{\text{HRD/q factor}} \ . \quad (1)$$

In the present TN I derive a few more relevant results which generalize this to

- estimate (the PDF over) absolute magnitude (section 3)
- add dependence on and estimation of the stellar metallicity (section 4)
- add dependence on and estimation of the surface gravity (section 5).

I also remind readers in section 2 of an important difference between conditional and unconditional independence.

Although first described in CBJ-049, the algorithm is described better in a MNRAS article, Bailer-Jones (2011), hereafter CBJ11. (Equation 1 is equation 15 in CBJ11.) There I also applied the algorithm to the estimation of temperature and extinction of 45 000 FGK stars using $BVJHK$ data and Hipparcos parallaxes. A few additional results are reported in Bailer-Jones (2011b). These articles are available from `http://www.mpia.de/homes/calj/qmethod.html`.

In the current TN I will use the notation in CBJ11, except that I will refer to the G-band rather than the V-band (so $M_{\rm G}$ rather than M_V, etc.) The notation is summarized in Table 1.

Note: the mathematics in this TN is not nearly as impenetrable as it first appears ...

2 Conditional dependence

I have discussed in general the significance and subtlety of conditional (in)dependence in CBJ-053 (section 4), but it is worth emphasizing something which is specific to the current problem.

The extinction parameter, A_0, is a property of the interstellar medium, and the effective temperatre, T, is a property of a star. They are, therefore, independent when they are not conditioned on other variables. That is

$$P(T|A_0) = P(T) \text{ and } P(A_0|T) = P(A_0) \quad \textit{unconditional independence} . \tag{2}$$

The situation can change if we now condition on a measurement.[1] The spectrum, $\boldsymbol{p}$, generally contains information on both variables. So it should be fairly obvious that

$$P(T|A_0, \boldsymbol{p}) \neq P(T) \quad \textit{conditional dependence} . \tag{3}$$

Less obvious, perhaps, is that in general

$$P(T|A_0, \boldsymbol{p}) \neq P(T|\boldsymbol{p}) \quad \textit{conditional dependence} . \tag{4}$$

The reason for this is that because $\boldsymbol{p}$ is informative about both variables, then additionally knowing A_0 changes what we know about T. That is, A_0 and T are unconditionally independent but conditionally dependent given $\boldsymbol{p}$. You can see this visually if you consider the "degeneracy plots" of the PDF $P(A_0, T|\boldsymbol{p})$, e.g. in Fig. 4 of CBJ11: If we fix (condition on) A_0, this changes what we know about T. Another way of appreciating this is to note that

$$P(T|\boldsymbol{p}) = \int_{A_0} P(T|A_0, \boldsymbol{p}) P(A_0|\boldsymbol{p}) dA_0 . \tag{5}$$

This does not equal $P(T|A_0, \boldsymbol{p})$, in general, because $P(A_0|\boldsymbol{p})$ is not uninformative (not a "flat" distribution in A_0).[2]

[1]The situation can also change if we introduce other information. For example, we may choose to introduce a Galactic model which says that both OB stars and high extinction regions tend to concentrate in the Galactic disk. In that case conditioning on Galactic latitude would introduce a dependence between A_0 and T.

[2]If it is uninformative then the two quantities are equal.

In summary, conditioning on a measurement can introduce a dependence between variables which were formerly independent. (Or if you prefer, conditioning can introduce a dependence in our *knowledge* or *estimates* of these variables.)

Note also that although $P(T|A_0, M_G) = P(T|M_G)$, $P(T|A_0, M_G) \neq P(T|A_0)$, because M_G and T are not independent: Varying the effective temperature of a star will, in general, change its absolute magnitude.

There are of course also cases where the opposite occurs: variables are unconditionally dependent but become independent when conditioned on another variable. The reader can no doubt think of examples (or can see section 4.2 of CBJ-053 for an example and a discussion).

3 Estimation of absolute magnitude

The quantity q is defined as

$$q \equiv G + 5\log\varpi = M_G + A_G - 5 . \tag{6}$$

q is measured, and once we have an estimate for A_G we could use this equation to estimate M_G. Yet this fails to take into account the known probability distributions over q and A_G, so may not provide an estimate of M_G which is consistent with our other parameter estimates or with the other information we have. The correct approach, as always, is to write down the probability distribution for M_G in terms of the measured quantities. Using Bayes' theorem and noting that $\boldsymbol{p}$ and q are independent, we have

$$\begin{aligned} P(M_G|\boldsymbol{p}, q) &= \frac{P(\boldsymbol{p}, q|M_G)\, P(M_G)}{P(\boldsymbol{p}, q)} \\ &= \frac{P(\boldsymbol{p}|M_G)\, P(q|M_G)\, P(M_G)}{P(\boldsymbol{p}, q)} . \end{aligned} \tag{7}$$

It's clear that we will have to marginalize over A_0 and T, as these terms are present in the q constraint and/or HRD prior. This motivates us to write the second term in the numerator as

$$P(q|M_G) = \int_{A_0,T} P(q|M_G, A_0, T)\, P(A_0, T|M_G)\, dA_0\, dT . \tag{8}$$

The first term in equation 8 is the familiar q constraint. As A_0 is independent of M_G, and T is independent of A_0, *when not conditioned on the data*, we can write the second term as

$$\begin{aligned} P(A_0, T|M_G) &= P(T|A_0, M_G)\, P(A_0|M_G) \\ &= P(T|M_G)\, P(A_0) \\ &= \frac{P(M_G, T)}{P(M_G)}\, P(A_0) . \end{aligned} \tag{9}$$

Equation 8 then becomes

$$P(q|M_G) = \int_{A_0,T} P(q|M_G, A_0, T)\, P(M_G, T)\, \frac{P(A_0)}{P(M_G)}\, dA_0\, dT . \tag{10}$$

In analogy to equation 8 we can also write $P(\boldsymbol{p}|M_{\rm G})$ as a marginalization

$$\begin{aligned} P(\boldsymbol{p}|M_{\rm G}) &= \int_{A_0,T} P(\boldsymbol{p}|M_{\rm G}, A_0, T)\, P(A_0, T|M_{\rm G})\, dA_0\, dT \\ &= \int_{A_0,T} P(\boldsymbol{p}|A_0, T)\, P(A_0, T|M_{\rm G})\, dA_0\, dT\;. \end{aligned} \tag{11}$$

The dependence on $M_{\rm G}$ has dropped out of the first term because by construction of the forward model A_0 and T entirely specify the spectrum. Using the result of equation 9 this becomes

$$P(\boldsymbol{p}|M_{\rm G}) = \int_{A_0,T} P(\boldsymbol{p}|A_0, T)\, P(M_{\rm G}, T)\, \frac{P(A_0)}{P(M_{\rm G})}\, dA_0\, dT \tag{12}$$

Putting equations 10 and 12 into equation 7 gives the final result

$$\begin{aligned} P(M_{\rm G}|\boldsymbol{p}, q) = & \underbrace{\frac{1}{P(M_{\rm G})\, P(\boldsymbol{p}, q)}}_{\text{priors}} \left(\int_{A_0,T} \underbrace{P(\boldsymbol{p}|A_0, T)}_{\text{likelihood}}\, \underbrace{P(M_{\rm G}, T)}_{\text{HRD prior}}\, \underbrace{P(A_0)}_{\text{prior}}\, dA_0\, dT \right) \\ & \times \left(\int_{A_0,T} \underbrace{P(q|M_{\rm G}, A_0, T)}_{\text{q constraint}}\, \underbrace{P(M_{\rm G}, T)}_{\text{HRD prior}}\, \underbrace{P(A_0)}_{\text{prior}}\, dA_0\, dT \right)\;. \end{aligned} \tag{13}$$

¡ Compared to the expression for $P(A_0, T|\boldsymbol{p}, q)$ (equation 1), here we also integrate over the likelihood, and overall there is now a squared dependence on the HRD and A_0 priors.

3.1 An approximation

Equation 13 is quite cumbersome and slow to evaluate numerically. We can make an entirely different approach to deriving $P(M_{\rm G}|\boldsymbol{p}, q)$ by involking the maximum entropy principle. This states that the least biased (or least informative) probability distribution given the available information is the one which maximizes the entropy (Jaynes 1957). Specifically, for a real-valued variable for which we only know the mean and variance, the maximum entropy distribution is a Gaussian.[3] In other words, if we simply estimate the expected value of $M_{\rm G}$ and its variance based on measurements of $\boldsymbol{p}$, q and/or parameters inferred from these, then it's conservative to assume that $P(M_{\rm G}|\boldsymbol{p}, q)$ has a Gaussian distribution. This is an approximation to the best estimate of the distribution given by equation 13 (which was derived using other information, so is not Gaussian). The mean may be taken as the value from equation 6,

$$M_{\rm G} = q - A_{\rm G} + 5 \tag{14}$$

where $A_{\rm G}$ is the best estimate of the extinction (e.g. derived from the mean or mode of $P(A_0|\boldsymbol{p}, q)$). Using the general result of variances (see equation A1 in CBJ11), the variance of $M_{\rm G}$ is

$$\begin{aligned} \sigma^2_{M_{\rm G}} = {\rm Var}(M_{\rm G}) = {\rm Var}(q - A_{\rm G} + 5) &= {\rm Var}(q) + {\rm Var}(A_{\rm G}) - 2{\rm Cov}(q, A_{\rm G}) \\ &= \sigma_q^2 + \sigma^2_{A_{\rm G}} - 2\rho(q, A_{\rm G})\sigma_q\sigma_{A_{\rm G}}\;. \end{aligned} \tag{15}$$

[3] This, incidentally, is why the Gaussian distribution is so ubiquitous for noise models: It's not because we really believe they are Gaussian, but rather because if we only know the mean (usually zero) and variance of the noise, then the Gaussian is the most conservative choice.

σ_q is given by the measurements and assumed noise models (see section 3.5 of CBJ11). σ_{A_G} can be calculated from the inferred PDF $P(A_0|\boldsymbol{p}, q)$. This gives σ_{A_0}, so we could apply a simple relationship to get σ_{A_G}.

$\rho(q, A_G)$ in equation 15 is the expected correlation between q and A_G *for any one star*. It should not be confused with the correlation between these variables for a sample of stars. To appreciate this distinction, consider the quantity $\rho(A_G, T)$. We know from earlier work that the estimates of A_G and T (for a single star) are highly correlated: the degeneracy can be see in Figure 4 of CBJ11. But whether or not these are correlated in any way for a sample of stars depends entirely on the sample. In Figure 14 of CBJ11 we see no such correlation over that sample. We could no doubt construct samples with different correlations, without affecting in any way the correlation between the estimates of the APs for any one star.

If we could reduce $\rho(q, A_G)$ into an expression in which the covariances are only between measured variables – variables for which we have a noise model – then we could estimate it for any specific star. But A_G has such a complicated dependence on $\boldsymbol{p}$ that there is no obvious simplification. (Using the full PDFs over the APs avoids having to make such simplifcations!) In lieu of this, the correlation could just be set to zero in equation 15.

4 Extension to metallicity

We saw in CBJ11 that the inference for A_0 and T was sensitive to the (distribution of) metallicity, Z, assumed in the HRD (i.e. it was a fixed prior). As we plan to estimate Z (or [Fe/H] if you prefer) from the BP/RP spectra, we need to generalize the model to include an explicit dependence on, and inference of, metallicity. This turns out to be almost identical to the derivation in section 2.7 of CBJ11.

From Bayes' theorem

$$P(A_0, T, Z|\boldsymbol{p}, q) = \frac{P(\boldsymbol{p}, q|A_0, T, Z)P(A_0, T, Z)}{P(\boldsymbol{p}, q)} . \quad (16)$$

Because $\boldsymbol{p}$ and q are independent measurements, and because A_0 and (T, Z) are unconditionally independent, this becomes

$$P(A_0, T, Z|\boldsymbol{p}, q) = \frac{P(\boldsymbol{p}|A_0, T, Z)\, P(q|A_0, T, Z)\, P(A_0)\, P(T, Z)}{P(\boldsymbol{p}, q)} . \quad (17)$$

The second term may again we written as a marginalization

$$P(q|A_0, T, Z) = \int_{M_G} P(q|M_G, A_0, T, Z)\, P(M_G|A_0, T, Z)\, dM_G . \quad (18)$$

As A_0 is independent of T and Z, then we can write the second term as

$$P(M_G|A_0, T, Z) = P(M_G|T, Z) = \frac{P(M_G, T, Z)}{P(T, Z)} . \quad (19)$$

Substituting equation 18 into equation 17 gives our final result

$$P(A_0, T, Z|\boldsymbol{p}, q) = \underbrace{P(\boldsymbol{p}|A_0, T, Z)}_{\text{likelihood}} \underbrace{\frac{P(A_0)}{P(\boldsymbol{p}, q)}}_{\text{priors}} \underbrace{\int_{M_G} \underbrace{P(q|M_G, A_0, T, Z)}_{\text{q constraint}} \underbrace{P(M_G, T, Z)}_{\text{HRD prior}} dM_G}_{\text{HRD/q factor}} \quad (20)$$

This is the same as equation 1 with T replaced by (T, Z). This is not surprising because Z has the same relationship as T toward A_0, M_G, $\boldsymbol{p}$ and q. The difference now is that the likelihood, HRD and q constraint all have to be generalized to include dependence on Z:

- To generalize the likelihood we have to extend the forward model to predict spectra also as a function of Z. This is straight forward enough, although we have to to pay special attention to the fact that Z is a weak parameter compared to A_0 and T. Sampling the likelihood becomes computationally more intense, as it is now a three dimensional function.
- Because $q \equiv G + 5\log\varpi$ is unchanged, the model for q is still $P(q|M_G, A_G)$. All we have to do is generalize the transformation of A_0 to A_G to include dependence on Z. Thus equation 19 in CBJ11 becomes $A_G = A_0 + y(A_0, T, Z)$. In practice the Z dependence of y is likely to be very weak, perhaps negligible.
- Generalizing the HRD to have a Z dependence is also straight forward, although if the sampling in Z is relatively sparse then we have to be careful how we smooth and/or sample it when estimating the posterior PDF.

In the above derivation we did not need to address the issue of the unconditional dependence of T and Z. But if we did, then for realistic stellar populations they are not independent, being linked by age: an old stellar population is dominated by cooler and metal poorer stars.

5 Extension to surface gravity

We now include estimation of $\log g$ to the original formulation by deriving an expression for $P(A_0, T, g|\boldsymbol{p}, q)$. Proceeding in a similar way as for the inclusion of Z and using the relevant independences we now get

$$P(A_0, T, g|\boldsymbol{p}, q) = \frac{P(\boldsymbol{p}|A_0, T, g)\, P(q|A_0, T, g) P(A_0)\, P(T, g)}{P(\boldsymbol{p}, q)} . \quad (21)$$

The first term is again the likelihood model for the photometric data, in which the forward model is now extended to predict $\log g$. For the second term, we must ask what information q has about $\log g$. By definition

$$g = \frac{\mathcal{G}m}{R^2} \quad (22)$$

($\mathcal{G}$ is the gravitational constant) and

$$L = 4\pi R^2 s T^4 \tag{23}$$

(s is Stefan's constant) and combining these we get

$$\log L = \log m + 4 \log T - \log g + \log(4\pi s\, \mathcal{G}) \; . \tag{24}$$

As L is closely related to $M_{\rm G}$, this establishes a relationship between $M_{\rm G}$, T, m and g. This tells us that in order to estimate g we need to introduce a dependency on all of these variables (just as in the original derivation of $P(A_0, T|\boldsymbol{p}, q)$ we introduced a dependence on $M_{\rm G}$). We therefore write down $P(q|A_0, T, g)$ as a marginalization over not just $M_{\rm G}$, but also m

$$P(q|A_0, T, g) = \int_{M_{\rm G}} \int_{m} P(q|M_{\rm G}, m, A_0, T, g)\, P(M_{\rm G}, m|A_0, T, g)\, dm\, dM_{\rm G} \; . \tag{25}$$

The first term in the integral can be simplifed to $P(q|M_{\rm G}, A_0, T)$, because q has no dependence on m or g once the other three variables are condtioned upon. Noting the lack of dependence on A_0 and using Bayes' theorem, the second term can be written

$$\begin{aligned} P(M_{\rm G}, m|A_0, T, g) &= P(M_{\rm G}, m|T, g) \\ &= \frac{P(M_{\rm G}, m, T, g)}{P(T, g)} \end{aligned} \tag{26}$$

Substituting these into equation 25 and that into equation 21 we get

$$\begin{aligned} P(A_0, T, g|\boldsymbol{p}, q) &= P(\boldsymbol{p}|A_0, T, g) \frac{P(A_0)}{P(\boldsymbol{p}, q)} \int_{M_{\rm G}} P(q|M_{\rm G}, A_0, T) \left[\int_{m} P(M_{\rm G}, m, T, g)\, dm \right] dM_{\rm G} \\ &= P(\boldsymbol{p}|A_0, T, g) \frac{P(A_0)}{P(\boldsymbol{p}, q)} \int_{M_{\rm G}} P(q|M_{\rm G}, A_0, T) P(M_{\rm G}, T, g)\, dM_{\rm G} \end{aligned} \tag{27}$$

where we have marginalized the four dimensional HRD $P(M_{\rm G}, m, T, g)$ over mass to get $P(M_{\rm G}, T, g)$. Comparing this to equation 1, we see that the equation is the same except with T replaced by (T, g).

Note that we have not made use of equation 24 in deriving equation 27. That equation implies that we could replace g by a relation $g = v(M_{\rm G}, m, T, {\rm BC})$ for some function v, where $\log L = M_{\rm G} + {\rm BC}$ and BC is the bolometric correction. But this would remove explicit dependence on g from the q-dependent part of the data, obviating what we are trying to achieve, namely using the parallax, magnitude and HRD prior to help estimate g. In practice one could nonetheless do this and then use such a relation to estimate g after having estimated $(M_{\rm G}, m, T)$. But we would no longer get a PDF over g, and we would loose the enforced consistency of this estimate with the one coming from the spectrum, $\boldsymbol{p}$.

5.1 The myth of the gravity and the parallax

It is often claimed that "measuring the parallax gives the surface gravity of the star" (assuming that you have estimated the apparent magnitude and effective temperature accurately). This statement is too simplistic and contains hidden assumptions. Equation 24 allows us to write the dependency of g on the other APs in a generic form $g = v(T, m, M_{\rm G}, {\rm BC})$. If we measure the parallax and the apparent magnitude, then these will only give us $M_{\rm G}$ if we already know $A_{\rm G}$, which, recall, is strongly degenerate with T in the BP/RP spectra (and in most other photometric estimates). That is the first complication. Second, the function v has a dependence on mass, but the mass is unknown. If we knew $M_{\rm G}$ and T accurately enough, then we would have some idea of mass (by assuming an HRD), but we cannot guarantee that it is constrained well enough to help in a determination of g. We could instead make use of a temperature–mass relation and effectively remove the explicit mass dependence, but such relations are neither accurate nor universal over the whole HRD. The third complication is the presence of the BC, which depends on the shape of the spectral energy distribution. This depends primarily on T, but not only, depending also on Z and on the (so far unknown) g. The fourth – but essentially trivial – complication is that all measures are noisy: if the parallax is not very accurate, it won't tell us anything about g.

So we see that there is no simple "deduction" of g once we measure the parallax. The quantities all hang together in complex way, and assumptions which might permit a "deduction" make assumptions about the very quantity we are trying to infer. The solution to this is to proceed in the way presented in this technical note: write down self-consistent, probabilistic equations for the dependence of the astrophysical parameters on all of the data and then marginalize over the unknown quantities. The accuracy of the g estimate we can get from this depends on the sensitivity of the spectrum to g, and how much information $P(M_{\rm G}, T, g)$ contains on g. It's the latter which helps the parallax "give" the gravity.[4]

6 Posterior PDF of all four APs (A_0, T, Z, g)

The extension to Z and g turned out to involve replacing T with (T, Z) and (T, g) respectively in equation 1 (something we may now like to claim as obvious with hindsight!). It should come as no surprise that the posterior PDF over all four main APs of interest is

$$P(A_0, T, Z, g|\boldsymbol{p}, q) = \underbrace{P(\boldsymbol{p}|A_0, T, Z, g)}_{\text{likelihood}} \underbrace{\frac{P(A_0)}{P(\boldsymbol{p}, q)}}_{\text{priors}} \underbrace{\int_{M_{\rm G}} \underbrace{P(q|M_{\rm G}, A_0, T)}_{\text{q constraint}} \underbrace{P(M_{\rm G}, T, Z, g)}_{\text{HRD prior}} \, dM_{\rm G}}_{\text{HRD/q factor}} \; . \tag{28}$$

The main differences with respect to equation 1 is that we now require a four dimensional forward model and a four dimensional HRD prior. Note that I have dropped the dependence

[4]It is vaguely amusing that some people are quite critical of the idea of using a prior in AP estimation, but five minutes latter will quite happily say that you can just use the parallax to determine $\log g$.

of the q contraint on both Z and g under the assumption that the conversion of A_0 to $A_{\rm G}$, $A_{\rm G} = A_0 + y(A_0, T, Z, g)$ can be approximated as $A_{\rm G} \simeq A_0 + y(A_0, T)$. If this approximation is not good enough, we can reintroduce the dependence on Z and/or g into the q constraint without changing anything else.

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