

# The Cosmic Triangle: Assessing the State of the Universe

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The cosmic triangle is introduced as a way of representing the past, present, and future status of the universe. Our current location within the cosmic triangle is determined by the answers to three questions: How much matter is in the universe? Is the expansion rate slowing down or speeding up? And, is the universe flat? A review of recent observations suggests a universe that is lightweight (matter density about one-third the critical value), is accelerating, and is flat. The acceleration implies the existence of cosmic dark energy that overcomes the gravitational self-attraction of matter and causes the expansion to speed up.

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As the new millennium approaches, novel technologies are opening new windows on the universe. Whereas previously we relied primarily on fossil evidence found in the local neighborhood of our galaxy to infer the history of the universe, now we can see directly the evolution of the universe over the past 15 billion years extending as far back as 100,000 years after the big bang. Thus far, the picture of the past history of the cosmos has altered only slightly: the new observations described in this paper are perfectly consistent with the standard big bang picture of the expansion of the universe from a hot, dense gas, the synthesis of the elements in the first few minutes, and the growth of structure through gravitational amplification of small, initial inhomogeneities. However, the expectation for the future has been dramatically revised. Based on the conventional assumption that the universe contains only matter and radiation – the forms of energy we can readily detect – the expectation for the future had been that the expansion rate of the universe would slow continuously due to the gravitational self-attraction of matter. The only major issue seemed to be whether the universe would expand forever or ultimately recollapse to a big crunch. Now, the mounting evidence described below is forcing us to consider the possibility that some cosmic dark energy exists that opposes the self-attraction of matter and is causing the expansion of the universe to accelerate.

Since the discovery of cosmic expansion by Hubble and Slipher [1] in the 1920's, the standard assumption had been that all energy in the universe is in the form of radiation and ordinary matter (electrons, protons, neutrons, and neutrinos – with mass counting as energy at the rate  $E = mc^2$ ). Over the next several decades, though, theory concerning the stability of galaxies [2], observations of the motion of galaxies in clusters [3, 4] and stars and gas surrounding galaxies [5, 6] indicated that most of the mass in the universe is dark and does not emit or absorb light [7, 8]. In the 1980's, the proposal of dark matter found resonance in the “inflationary universe” scenario [9, 10, 11, 12], a theory of the first  $10^{-30}$  seconds designed to address

several questions left unanswered by the big bang model: Why is the universe so homogeneous and isotropic? Why is the curvature of space so insignificant? Where did the initial inhomogeneities come from that give rise to the formation of structure [13, 14, 15, 16]? The standard inflationary theory predicts that the universe is spatially flat; according to Einstein’s theory of general relativity, this fixes the total energy density of the universe to equal precisely the critical value,  $\rho_c \equiv 3H_0^2/8\pi G \sim 1.7 \times 10^{-29} \text{ g cm}^{-3}$ , where  $H_0$  is the current value of the Hubble parameter and  $G$  is Newton’s gravitational constant (see Eq. (1)). Measurements show that ordinary matter and radiation account for less than 10% of the predicted value [17, 18, 19]. Inflation thus seemed to call for dark matter. The observational case for dark matter continued to grow, as discussed below, and particle physicists proposed various hypothetical particles, motivated by supersymmetry and unified theories, that could reasonably explain the dark matter. The new consensus model became the “cold dark matter” picture which predicts that the universe contains primarily cold, nonbaryonic dark matter [20, 21, 22]. Although the total mass density identified by observations still fell short of the critical value [7, 8, 23, 24], many cosmologists adopted as a working hypothesis the critical density model, trusting that something would fill the gap.

The last few years have seen signs of another shake-up of the standard model [25, 26, 27]. First, improved observations confirmed that the total mass density is probably less than half of the critical density [28, 29, 30]. At the same time, combined measurements of the cosmic microwave background (CMB) temperature fluctuations and the distribution of galaxies on large scales began to suggest that the universe may be flat [26], consistent with the standard inflationary prediction. The only way to have a low mass density and a flat universe, as expected from the inflationary theory, is if an additional, nonluminous, “dark” energy component dominates the universe today. The dark energy would have to resist gravitational collapse, or else it would already have been detected as part of the clustered energy in the

halos of galaxies. But, as long as most of the energy of the universe resists gravitational collapse, it is impossible for structure in the universe to form. The dilemma can be resolved if the hypothetical dark energy was negligible in the past and then over time became the dominant energy in the universe. According to general relativity [31], this requires that the dark energy have a remarkable feature: *negative* pressure. This argument [26] would rule out almost all of the usual suspects, such as cold dark matter, neutrinos, radiation, and kinetic energy, because they have zero or positive pressure.

With the recent measurements of distant exploding stars, supernovae, the existence of negative-pressure dark energy has begun to gain broad consideration. Using Type Ia supernovae as standard candles to gauge the expansion of the universe, observers have found evidence that the universe is accelerating [33, 34]. A dark energy with significant negative pressure [26, 32] will in fact cause the expansion of the universe to speed up, so the supernova observations provide empirical evidence of a dark energy with strongly negative pressure [33, 35, 36, 37].

The news has brought the return of the cosmological constant, first introduced by Einstein for the purpose of allowing a static universe with the repulsive cosmological constant delicately balancing the gravitational attraction of matter [38]. In its present incarnation, the cosmological constant is out of balance, causing the expansion of the universe to accelerate. It can be viewed as a vacuum energy assigned to empty space itself, a form of energy with negative pressure. Cosmologists are familiar with other hypothetical forms of dark energy with negative pressure that can accelerate the universe. In inflationary cosmology, a cosmic field, similar to an electric field in the sense that it pervades space and assigns a field value and energy to each point in it; acceleration is caused by a cosmic field whose kinetic energy is much less than its potential energy [10, 11]. A different field, with much tinier energy, coined “quintessence” [39], could account for the acceleration suggested to be observed today. Unlike a cosmological constant, quintessence energy changes with time and naturally develops inhomogeneities that can produce

variations in the distribution of mass and the CMB temperature observed today [39, 40].

The introduction of a dark energy component which leads to the acceleration of the universe is neither a simple nor modest change. New challenges to cosmology and to fundamental physics are immediately posed. And the future of the universe is totally altered. Although it is premature to call the new evidence for an accelerating expansion and exotic dark energy conclusive, the case is strong enough and the implications dramatic enough that the time is ripe for a new report on the state of the universe.

### The Cosmic Triangle

According to Einstein's theory of general relativity, the evolution of the universe is determined by the forms of energy it contains and the curvature of space. Einstein's equations can be reduced to a simple form known as the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \quad (1)$$

On the left-hand-side is the Hubble parameter  $H = H(t)$ , which measures the expansion rate of the universe as a function of time. The current value,  $H_0$ , is  $65 \pm 10 \text{ km sec}^{-1} \text{ Mpc}^{-1}$ , where one megaparsec (Mpc) is  $3.26 \times 10^6$  light years [41, 42, 43]. The right-hand side contains the factors which determine the expansion rate. The first factor is the energy density  $\rho$  (times Newton's gravitational constant  $G$ ). The energy density  $\rho = \rho(t)$  can have several different subcomponents: a mass density associated with ordinary and dark matter, the kinetic energy of the particles and radiation, the energy associated with fields (such as quintessence), and the vacuum energy density or, equivalently, the cosmological constant.

The second term on the right-hand side describes the effect of curvature of space on the expansion of the universe. The curvature constant  $k$  can be positive, negative or zero. The parameter  $a = a(t)$ , known as the scale factor, measures how much the universe stretches as a function of time. It can be

thought of as proportional to the average distance between galaxies. As the universe stretches, the curvature is diminished, as indicated in the equation. The terms closed, open and flat refer, by definition, to the cases of positive, negative, and zero curvature. It has been common to use the same terms to describe whether the universe will ultimately recollapse, expand forever, or lie on the border between expansion and recollapse. This second usage does not necessarily apply if there is vacuum density or quintessence, a point which often causes confusion. For example, if there is vacuum energy, it is possible to have a universe that is closed (positive curvature), but which expands forever because the acceleration due to the cosmological constant overcomes the curvature effect [44] which would otherwise bring the expansion to a halt and then recollapse.

For simplicity, we will consider a universe composed today of baryonic (ordinary) and dark (exotic) matter, curvature and vacuum energy (*i.e.*, a cosmological constant,  $\Lambda$ ). The fractional contributions to the right-hand side of the Friedmann equation, which depend on the relative values of the matter density, vacuum energy density ( $\rho_\Lambda$ ) and curvature, are given the symbols  $\Omega_m \equiv 8\pi G\rho_{\text{matter}}/(3H^2)$ ,  $\Omega_\Lambda \equiv 8\pi G\rho_\Lambda/(3H^2) \equiv \Lambda/(3H^2)$ , and  $\Omega_k \equiv -k/(aH)^2$ , respectively [45]. Dividing both sides of Eq. (1) by  $H^2$  yields a simple sum rule

$$1 = \Omega_m + \Omega_k + \Omega_\Lambda. \quad (2)$$

For any other energy component, such as quintessence, a term  $\Omega_Q$  would be added to the right-hand side of Eq. (2). The sum rule can be represented by an equilateral triangle (Fig. 1). Lines of constant  $\Omega_m$ ,  $\Omega_k$  and  $\Omega_\Lambda$  run parallel to each of the edges of the equilateral triangle. Every point lies at an intersection of lines of constant  $\Omega_m$ ,  $\Omega_k$  and  $\Omega_\Lambda$  such that the sum rule is satisfied. Although  $\Omega_m$  is non-negative, the curvature and cosmological constant can be positive or negative.

Inflationary theory [9, 10, 11, 12] proposes that the universe underwent

a brief epoch of extraordinary expansion during the first  $10^{-30}$  seconds after the big bang which ironed out the curvature, setting  $\Omega_k = 0$ . If the curvature is zero, i.e., the universe is flat, then the sum rule reduces to  $\Omega_m + \Omega_\Lambda = 1$ , corresponding to the blue line (marked “flat”) in the figure. The yellow line indicates the division between models in which the expansion rate is currently decelerating versus accelerating. The competition between the decelerating effect of the mass density and the accelerating effect of the vacuum energy density can be understood from Einstein’s equation for the stretching of the scale factor  $a(t)$ :

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a, \quad (3)$$

where  $p$  is the pressure associated with whatever energy is contained within the universe. If the universe contains ordinary matter and radiation, then  $\rho + 3p$  is positive, and the expansion decelerates,  $\ddot{a} < 0$ . However, exotic components like vacuum-energy and quintessence [39, 40] have sufficient *negative* pressure to make  $\rho + 3p$  negative, inducing cosmic acceleration.

Finally, models of special interest have been highlighted in the Figure; they form nearly an equilateral triangle of their own. The standard cold dark matter model (SCDM), the most simple possibility, has  $\Omega_m = 1$  and no curvature or vacuum component. The model assumes a “scale-invariant” spectrum of initial density fluctuations, a spectrum in which the magnitude of the inhomogeneity is the same on all length scales, as predicted by standard inflationary cosmology [13, 14, 15, 16]. A model that better fits the observations and which retains the simple condition of  $\Omega_m = 1$  is the “tilted” TCDM in which the fluctuation spectrum is tilted so that the average inhomogeneity increases with length scale, unlike the standard inflationary prediction. The open cold dark matter (OCDM) model has low mass density and no vacuum component; the best-fit version has a mixture of one-third matter density and two-thirds curvature (but no vacuum energy) with a spectrum that is tilted the opposite way from the TCDM model (the inhomogeneity decreases as the length scale increases). As an example of a dark energy

plus cold dark matter model ( $\Lambda$ CDM), a current best estimate model, we will consider a mixture of two-thirds vacuum density (or cosmological constant,  $\Lambda$ ) and one-third matter density (but no curvature) and the standard un-tilted spectrum predicted by inflationary theory. The parameters for each of the four models, shown in Table I, has been chosen to best fit for each type of model to the current observational constraints discussed below. All the models (and our analysis) assume the standard inflationary prediction that the density fluctuations are gaussian and adiabatic (*i.e.*, radiation, ordinary and dark matter fluctuate spatially in the same manner) [13, 14, 15, 16], which agree with current observations. The age of the universe (in units of  $10^9$  y) is consistent with the most recent estimates of the ages of the oldest stars [46, 47].

The models may be distinguished observationally by answering three fundamental questions: Is there enough matter to close (flatten) the universe? Is the expansion rate accelerating, providing evidence for a new dark energy? Is the universe curved? In the next sections we describe a series of independent tests aimed at addressing these three questions. The best constraint for each question is represented as a strip in the plot in Fig. 2. Together, these constraints determine our location in the cosmic triangle plot and, thereby, the past and future evolution of the universe.

### **Is There Enough Mass to Close the Universe?**

The consensus for a low-mass-density universe ( $\Omega_m < 1$ ) has been building slowly for over a decade, although, truth be said, there was never credible evidence otherwise [7, 8, 23, 24, 26, 28, 29, 30]. The determination of the universe's mass density is currently the best-studied of the three cosmological parameters, and is supported by a number of independent measurements. Although each observation has its strengths, weaknesses and assumptions, they all indicate that  $\Omega_m < 1$ .

*Mass-to-light method.* One of the oldest and simplest techniques for estimating the total mass of the universe entails a two-step process: first determine



the average ratio of the mass to the emitted light of the largest systems possible; then, multiply by the total measured luminosity density of the universe. This totals up all the mass associated with light to the largest scales. Rich clusters of galaxies are the largest (1-2 Mpc in radius) and most massive ( $2 - 10 \times 10^{14}$  solar masses within 1 Mpc) bound systems known for which mass has been reliably measured. Nowadays, cluster mass can be inferred from three independent methods: the galaxy motion within the cluster, the temperature of the hot intracluster gas, and gravitational lensing by the cluster mass (the distortion of background galaxies' images by the cluster's gravitational potential). There is good agreement among these independent estimators. The mean cluster mass-to-light ratio ( $M/L$ ), about  $200 \pm 70$  times the mass-to-light ratio for the sun, indicates that there is a great deal of dark matter within clusters [28, 29]. Nevertheless, even if we assume that all light in the universe is emitted from objects that have as much mass per unit light as clusters, the total mass would not be sufficient to close the universe. Multiplying  $M/L$  by the observed luminosity density, one obtains  $\Omega_m = 0.2 \pm 0.1$  [7, 28, 29]. Recent studies of the dependence of  $M/L$  on scale indicate that  $M/L$  is nearly constant on large scales ranging up to supercluster size (10 Mpc), suggesting no additional dark matter is tucked away on large scales [28, 48].

*Baryon fraction method.* An independent method of estimating the mass of the universe, also based on rich clusters, entails measuring the ratio of the baryonic to total mass in clusters [49, 50]. Because clusters form through gravitational collapse, they scoop up the mass over a large volume of space such that the ratio of baryons to total matter in the collapsed cluster should be representative of the cosmic average to within 20% [51, 52]. The big bang model of primordial nucleosynthesis constrains the baryon density to be  $\Omega_b = 0.045 \pm 0.0025$  (based on the cosmic abundance of helium and deuterium, and using  $H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ) [17, 18, 19]. Thus, if one can measure the average baryon ratio in the universe,  $\Omega_b/\Omega_m$ , it can be used with the

known  $\Omega_b$  to determine  $\Omega_m$ . A cluster's baryon ratio can be determined from the baryonic mass in the cluster (obtained by measuring the x-ray emission from the hot intracluster gas and adding the mass of the stars [49, 53]) divided by the total cluster mass. The baryon ratio is found to be  $\Omega_b/\Omega_m \approx 0.15$ , much larger than the 0.045 value expected if  $\Omega_m = 1$  [49, 50, 51, 52]. The observed ratio corresponds to a mass density of  $\Omega_m = 0.3 \pm 0.1$ . If some baryons are ejected from the cluster during gravitational collapse, as suggested by cosmological simulations [51, 52], or if some baryons are bound in nonluminous objects such as rocks or planetary-sized objects, then the actual value of  $\Omega_m$  is lower than this estimate.

*Cluster Abundance and Its Evolution.* A third feature of rich clusters which constrains  $\Omega_m$  is the number density of clusters as a function of cosmic time (or red shift) [30, 54, 55, 56, 57, 58]. Rich clusters are the most recently formed gravitationally bound objects in the universe. The observed present-day ( $z \sim 0$ ) cluster abundance provides a strong constraint on the normalization of the power spectrum of density fluctuations— the seeds that created the clusters— on the relevant cluster scales (see next subsection) [24, 54, 59]. The  $\Lambda$ CDM and OCDM models are consistent with the observed cluster abundance at  $z \sim 0$ . SCDM, however, when normalized to match the observed fluctuations in the CMB (see next subsection), produces too many clusters at all red shifts (Fig. 3) [24, 55, 59, 60]. The TCDM model preserves  $\Omega_m = 1$  and more nearly fits the present day cluster abundance (Fig. 3, 4).

The evolution of cluster abundance with red shift breaks the  $z = 0$  degeneracy among the models [30, 54, 55, 56]. The  $\Omega_m < 1$  models ( $\Lambda$ CDM and OCDM) predict relatively little change in the number density of rich clusters as a function of red shift because, due to the low matter density, hardly any structure growth has occurred since  $z \sim 1$ . For the  $\Omega_m = 1$  TCDM model, structure has been growing steadily and rich clusters could only have formed recently; the number density of rich clusters at  $z \simeq 0.5 - 1$  is predicted to be exponentially smaller than today. The observation of even one massive

cluster at high red shift ( $z > 0.6$ ) suffices to rule out the  $\Omega_m = 1$  model. In fact, three clusters have been observed already (Fig. 3), suggesting a low-density universe,  $\Omega_m = 0.25_{-0.10}^{+0.15}$  ( $1\sigma$ ) [30]. A caveat for this method is that it assumes that the initial spectrum of density perturbations is gaussian, as predicted by inflation, which has not yet been carefully confirmed observationally (but see [61]) on the cluster scales.

*Mass Power Spectrum.* The mass power spectrum (Fig. 4) measures the degree of inhomogeneity in the universe's mass distribution on different distance scales. Beginning from a cosmological model, the mass power spectrum depends on the initial spectrum of inhomogeneities (*e.g.*, the stretched-out quantum fluctuations predicted by inflation), the recent creation of new perturbations, and how those inhomogeneities have evolved over time (which depends on the cosmological parameters). Existing measurements of the present day abundance of galaxy clusters constrain the mass inhomogeneity on the smallest scale for which the power spectrum can be reliably interpreted (about 10 Mpc). Observations of temperature fluctuations in the cosmic microwave background across the sky, as measured by the COBE satellite [62], constrain both the amplitude and shape of the spectrum on the largest observable scales.

Galaxy surveys are beginning to probe intermediate scales, from 10 to 1000 Mpc (Fig. 4) [63, 64]. The theoretical model predicts the distribution of all the mass, while observations of galaxies reflect the luminous, baryonic matter only. If the luminous matter follows the total mass, the mass distribution is said to be "unbiased." Otherwise, the ratio of overdensity in luminous matter to that in the total mass is termed the "bias." On small scales, the bias may vary with distance scale and local environment. These complications can be avoided by focusing on measurements of the power spectrum on large scales (more than 10 Mpc; Fig. 4) where the inhomogeneities are small and the bias is expected to be small [65, 66]. Although current galaxy measurements are inconclusive, especially given uncertainty in the bias, fu-

ture surveys, such as the Sloan Digital Sky Survey [64], are poised to test the shape of the spectrum on the intermediate scales.

Inflation predicts the *shape* of each spectrum [13, 14, 15, 16], but it does not predict its *normalization* (i.e., amplitude). The normalization is determined from observations, mainly the observed cluster abundance (on 10 Mpc scales) and the CMB fluctuations (on 1000 Mpc scales). The  $\Omega_m = 1$  SCDM model, normalized to the CMB fluctuations on large scales, is inconsistent with the cluster abundance (predicting over ten times more clusters than observed; Fig. 3). SCDM is thus inconsistent with observations [24, 26, 30, 54, 57, 59, 60]. The model can be “forced” to agree approximately with both the cluster abundance on small scales and the CMB fluctuations on large scales by tilting the power spectrum (by about 30%) from its standard shape. This tilted variant of the SCDM model—TCDM—is thus nearly consistent with both constraints. The power spectra of the  $\Lambda$ CDM and OCDM models can be normalized so that they agree with both the CMB and cluster observations (with a 30% tilt needed for OCDM). Future observations, on all scales, will greatly improve the power spectrum constraints. This will allow a measure of  $\Omega_m$  from the shape of the spectrum; currently this measure suggests a low value of  $\Omega_m$ , but with large uncertainty.

*Overall Estimate of  $\Omega_m$ .* The independent methods described above using clusters of galaxies yield a consistent determination of a low mass-density universe, nearly independent of  $\Omega_\Lambda$  or  $\Omega_k$ . Other methods discussed later in the paper, such as statistics of gravitational lensing, large scale velocities [67], and measurements of the CMB anisotropy, place additional constraints on  $\Omega_m$  in combination with other parameters and assumptions; though less constraining, these too suggest a low-mass density [36, 37]. The net result is represented by the “clusters” band ( $1\sigma$ ) in the cosmic triangle of Fig. 2. It is remarkable that a single value of  $\Omega$ ,  $\Omega_m \simeq 1/3$ , is consistent with so many, diverse observations.

### Is the Universe’s Expansion Accelerating?

Changes in the cosmic expansion rate can be studied using the observed brightness-red shift relation. A set of standard candles (objects of known luminosity) spread throughout the universe are used to determine the relation between distance and red shift. The distance  $d_L$  is determined by comparing the known luminosity  $L$  to the flux observed at Earth,  $f_{obs}$ , and invoking the inverse-square law ( $f_{obs} = L/4\pi d_L^2$ ). By studying standard candles at different observed fluxes, we study objects whose light was emitted at different cosmic times. The red shift  $z$  of the object measures the expansion of the universe since that time.

For relatively nearby standard candles the distance  $d_L$  is a simple linear function of red shift, as given by the Hubble relation of the expanding universe:  $H_0 d_L = cz$ , where  $c$  is the speed of light. However, the linear relation is only an approximation. If we study standard candles farther away, the non-linearities in the  $d_L$ - $z$  relation become important because the universe's expansion may be decelerating or accelerating. The results are most sensitive to the difference between  $\Omega_m$  (which decelerates the expansion) and  $\Omega_\Lambda$  (which accelerates the expansion) and are rather insensitive to the curvature  $\Omega_k$ .

*Supernovae.* Type Ia supernovae are the current best candidates for standard candles. They have the advantage that they are bright and can be seen at cosmic distances. As a class, Type Ia supernovae are not *all* identically luminous, but examination of nearby supernovae indicates that they may be converted into reliable distance indicators by calibrating them according to the time scale of their brightening and fading [68, 69]. Two efforts are underway to collect data on the red shift, luminosity and light curves of distant supernovae, the Supernova Cosmology Project (SCP) [33, 69, 70] and the High-Z Supernova Search (HZS) [34, 71, 72]. By now, studies of over 50 Type Ia supernovae at  $z = 0.3$  to  $0.9$  have been published and calibrated with a comparable number of nearby supernovae [73, 74] at  $z \lesssim 0.1$ .

The results of the two studies show that the distant supernovae are fainter, thus more distant than expected for a decelerating universe (Fig. 5). It

appears that the expansion rate is accelerating, indicating the existence of dark energy with negative pressure, such as  $\Omega_\Lambda$ . The best-fit results (Fig. 2) can be approximated by the linear combination  $0.8\Omega_m - 0.6\Omega_\Lambda = -0.2 \pm 0.1(1\sigma)$  [33, 75]. For a flat universe ( $\Omega_m + \Omega_\Lambda = 1$ ), the best-fit values are approximately  $\Omega_m = 0.25 \pm 0.1$  and  $\Omega_\Lambda = 0.75 \pm 0.1$  ( $1\sigma$ ) for the combined results of SCP team [33] and the two analyses of the HZS team [34]. These values are in excellent agreement with the  $\Omega_m$  results discussed above. In particular, all flat  $\Omega_m = 1$  models, which are identical in their  $d_L - z$  predictions, are formally ruled out at the  $8\sigma$  level.

The caveats for this test are possible uncertainties in the cross-comparison of the near and distant supernovae. Distant supernovae are calibrated using nearby supernovae assuming that the lightcurve time scale accounts for any relevant evolution of Type Ia supernovae. Although known evolutionary and dust obscuration effects have been taken into account, there remains the concern that there are additional evolutionary or dust effects at large red shift that have not been noted before. Further investigations are underway using observations comparing nearby and distant supernovae. The current results suggest that the expansion of the universe is accelerating, indicating the existence of a cosmological constant or dark energy.

*Gravitational Lensing Statistics.* Gravitational lensing due to accumulations of matter along the line of sight to distant light sources provide another potentially sensitive measure of our position in the cosmic triangle. These measures can be used in two ways. The first method uses the abundance of multiply imaged sources such as quasars, lensed by intervening galaxies [76, 77, 78, 79]. The probability of finding lensed images is directly proportional to the number of galaxies (lenses) along the path and thus to the distance in light-years to the source. This distance (for fixed  $H_0$ ) increases dramatically for a large value of the cosmological constant: the age of the universe and the distance to the galaxy become large in the presence of  $\Omega_\Lambda$  because the universe has been expanding for a longer time (compared with an  $\Omega_m = 1$  case); therefore,

more lenses are predicted if  $\Omega_\Lambda > 0$ . Using this method, an upper limit of  $\Omega_\Lambda < 0.75$  (95%CL) has been obtained [76, 77, 78, 79], marginally consistent with the supernovae results. The caveats of this powerful method include its sensitivity to uncertainties in the number density and lensing cross-section of the lensing galaxies and the number density of distant faint quasars. A second method is lensing by massive clusters of galaxies [80, 81]. Such lensing produces widely separated lensed images of quasars and distorted images of background galaxies. The observed statistics of this lensing, when compared with numerical simulations, rule out the  $\Omega_m = 1$  models [80, 81] and set an upper bound of  $\Omega_\Lambda < 0.7$  [81]. The limit is sensitive to the resolution of the numerical simulations, which are currently improving.

### Is the Universe Curved?

The curvature of the universe can be measured from the highest-red shift cosmological test, the cosmic microwave background. The CMB power spectrum provides a measure of the inhomogeneity in matter and energy at  $z \approx 1000$ , corresponding to a few 100,000 years after the big bang. The power spectrum is the root-mean-square fluctuation in the CMB temperature—the temperature “anisotropy”—as a function of the angular scale expressed as an integer multipole moment,  $\ell$ . A given  $\ell$  corresponds roughly to angle  $\pi/\ell$  radians. Each cosmological model produces a distinguishable CMB temperature anisotropy fingerprint [82, 83]. On large angular scales (small  $\ell$  values), the CMB spectrum probes inhomogeneities which span distances so large ( $\sim 1000$  Mpc) that neither light nor any other interaction has had time to traverse or modify them. These inhomogeneities are a direct reflection of the initial spectrum (*e.g.*, as created by inflation). If the models predict an un-tilted or a tilted spectrum, then the CMB anisotropy spectrum has a plateau that is flat or tilted, respectively. On small angular scales (less than a degree or  $\ell > 200$ ), the anisotropy spectrum has peaks and valleys created by the small-scale inhomogeneities; on these scales, there has been sufficient time for light to traverse them and for the matter to respond gravitationally to

the density fluctuation. The hot gas of baryons and radiation begin a series of acoustic oscillations in which matter and radiation are drawn by gravity into regions of high density and then rebound due to the finite pressure of the gas. On scales corresponding to the “sound horizon” (the maximum distance pressure waves can travel from the beginning of the universe up to the time the CMB is emitted), the mass has had time to undergo maximum collapse around the dense regions so as to produce maximum anisotropy but has not had time to rebound. Hence, a peak in the power spectrum is anticipated on the angular scale corresponding the sound horizon, and this should be the peak with the largest angular scale (smallest  $\ell$ ). An interesting feature is that the physical length corresponding to the sound horizon is relatively insensitive to the cosmological model. The angular scale which it subtends on the sky depends only on the overall curvature of space; the curvature distorts the path of light so that the sound horizon appears bigger or smaller on the sky depending on whether the curvature is positive or negative. If the universe is flat, the sound horizon subtends about a degree on the sky (resulting in a power spectrum peak near  $\ell \approx 200$ ), whereas the angular size is smaller in a curved open model (resulting in a peak near  $\ell \approx 200/\sqrt{\Omega_m + \Omega_\Lambda}$ ) [84].

The CMB anisotropy was first detected by the COBE (Cosmic Background Explorer) satellite in 1992 [85], followed by a series of ground- and balloon-based experiments [87, 88, 89, 90, 91]. Here we have selected published experiments which measure at several frequencies (to eliminate foreground sources) and which have been cross-correlated with other measurements (Fig. 6). In the next few years, there will be a sequence of ground- and balloon-based experiments culminating in the NASA MAP (Microwave Anisotropy Probe) and the ESA PLANCK satellite missions, which will produce all-sky temperature maps with a few arcminutes resolution. These improved maps will do much more than measure the position of the first acoustic peak and, thereby, the curvature; by measuring the detailed shape of the plateau and a sequence of peaks with very high precision, they will confirm (or refute) the basic underlying cosmological scenario and, if con-



firmed, will help to determine additional cosmological parameters, such as  $\Omega_m$ ,  $\Omega_\Lambda$ ,  $\Omega_b$ ,  $H_0$ , and more [92, 93]. The best-fit parameter region derived from the current CMB results shown in Fig. 2, is consistent with a flat universe, although the uncertainty is large. This analysis assumes that the initial fluctuations are adiabatic, as predicted by the standard inflationary theory and as assumed in our four models (see discussion above). If they are not this will be apparent from future CMB observations, and a different means can be used to extract the curvature from CMB data.

If the universe is flat and the matter density is less than the critical density, then there must be some form of nonclustering dark energy. In that case, as discussed in the introduction, the only way to form the observed large-scale structure is if its pressure is negative, because that guarantees that its density was negligible in the past when structure formed. This conclusion is consistent with the evidence suggesting that the universe is accelerating, which can only occur with a substantial negative pressure component.

### **The Cosmic Triangle: Present, Past and Future**

The current state of the universe can be surmised from the answers to the three questions posed above. The most precise measurements of the mass (using clusters), the acceleration (using supernovae), and the curvature (using the CMB) each confine the universe to a strip in the cosmic triangle plot (Fig. 2). *All three strips overlap at the  $\Lambda$ CDM model with approximately  $\Omega_m = 1/3$ ,  $\Omega_\Lambda = 2/3$ , and  $\Omega_k = 0$  [36, 37].* Zero curvature is consistent with inflation.

The verification and refinement of these conclusions will take place in the next few years through experiments already underway and will finally settle some of the questions that have challenged cosmologists for most of the 20th century. However, new cosmological challenges will take their place. Establishing inflation as the source of the fluctuations that seeded galaxy formation requires tests of the shape, gaussianity, and gravitational wave component

of the primordial power spectrum [82, 94, 95, 96]. As estimates of the cold dark matter density become more precise, it becomes even more imperative that its composition be identified. A host of candidates are suggested by particle physics models [97]. The leading candidates at present are the axion [98] and the lightest, stable, supersymmetry partner particles, such as the photino and higgsino [99]. (Recent measurements of atmospheric and solar neutrinos show that the neutrino has a small mass, but the mass is probably too small to be important cosmologically [100].)

However, it is the acceleration of the universe that raises the most provocative and profound issues. The acceleration may be caused by a static, uniform vacuum density (or cosmological constant) or by a dynamical form of evolving, inhomogeneous dark energy (quintessence) [39, 40]. Distinguishing between the two cosmologically is important because it informs us of what kind of new fundamental physics is required to explain our universe. Promising approaches include measurements of supernovae, CMB anisotropy and gravitational lensing [33, 35, 36, 37]. Special initial conditions are required for the vacuum energy possibility because it remains constant while the matter density decreases over 100 orders of magnitude as the universe expands. In order to have a vacuum energy density only a factor of two greater than the matter density today, it would have to have been exponentially small compared to the matter density in the early universe. A major motivation for proposing quintessence is that its interactions can cause its energy to naturally adjust itself to be comparable to the matter density today without special initial conditions [101].

Acceleration also affects our projection for the future fate of the universe, which can also be represented in a cosmic triangle plot (Fig. 7). As the universe evolves,  $\Omega_m$ ,  $\Omega_k$  and  $\Omega_\Lambda$  change at different rates, while maintaining a total value of unity, according to the sum rule. Possible trajectories to the future (Fig. 7) show that  $\Omega_m = 1$  is an unstable fixed point and  $\Omega_\Lambda = 1$  is a stable fixed point. If  $\Omega_m < 1$  today and there is any bit of added dark energy, then we are ultimately careening towards a flat  $\Omega_m \rightarrow 0$  ( $\Omega_\Lambda \rightarrow 1$ ) universe

in which the matter is spreading infinitesimally thinly leaving behind only an inert vacuum energy. If the vacuum energy (or quintessence) is unstable this fate may be averted.

As the current millenium ends, the past history and the present state of the universe are making themselves known. Determining the long-term fate of the universe will require an understanding of the fundamental physics underlying the dark energy, one of the grand challenges for the millenium to come.

Table 1: The basic parameters for the four models considered in this paper:  $\Omega_m$ ,  $\Omega_\Lambda$ , and  $\Omega_k$  are the ratio of the mass, vacuum energy and curvature to the critical density.  $H_0$  is the Hubble parameter in  $\text{km s}^{-1} \text{Mpc}^{-1}$ . The tilt measures how the amplitude of the inhomogeneity in initial density perturbations changes with length scale; tilt equal to unity means that the amplitude is scale-invariant, the inflationary prediction. The last column is the age of the universe in billions of years. The first model,  $\Lambda\text{CDM}$ , is in best agreement with observations (see Fig. 2).

Models and Parameters

Model	$\Omega_m$	$\Omega_\Lambda$	$\Omega_k$	$H_0$	tilt	age
Cosmological Const. ( $\Lambda\text{CDM}$ )	1/3	2/3	0	65	1	14.1
Open (OCDM)	1/3	0	2/3	65	1.3	12.0
Standard (SCDM)	1	0	0	50	1	13.0
Tilted (TCDM)	1	0	0	50	0.7	13.0

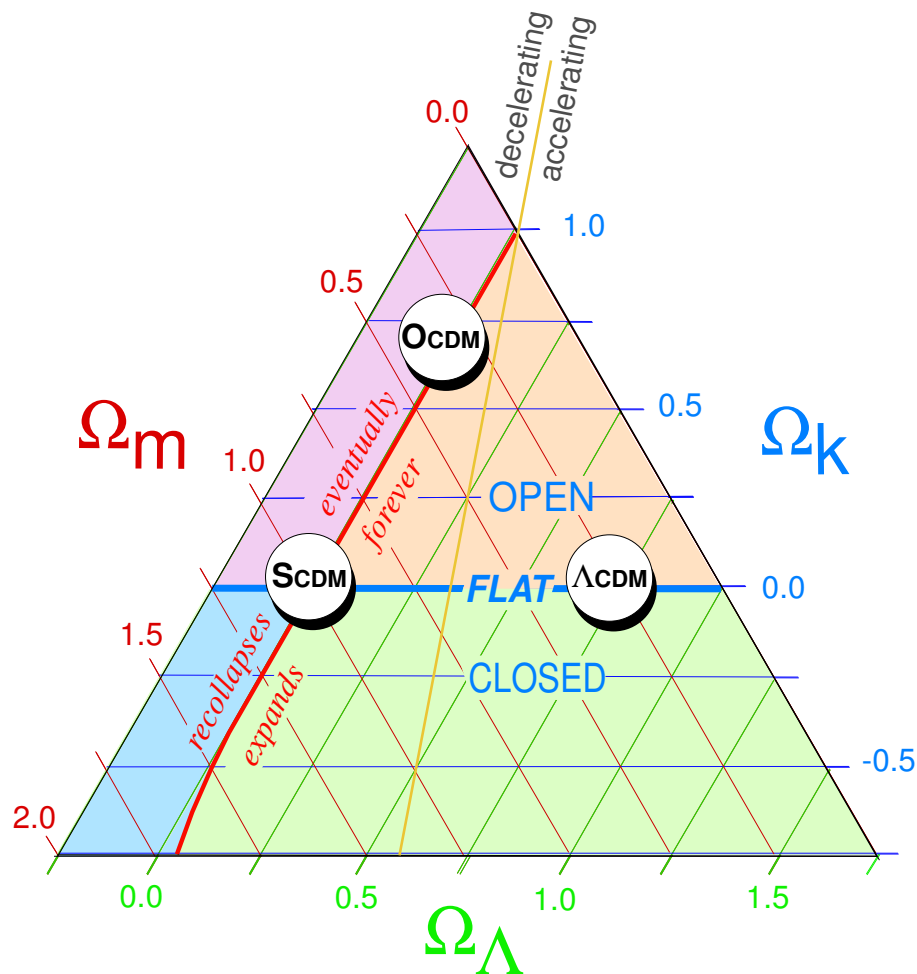


Figure 1: **The Cosmic Triangle** represents the three key cosmological parameters –  $\Omega_m$ ,  $\Omega_\Lambda$ , and  $\Omega_k$  – where each point in the triangle satisfies the sum rule  $\Omega_m + \Omega_\Lambda + \Omega_k = 1$ . The blue horizontal line (marked Flat) corresponds to a flat universe ( $\Omega_m + \Omega_\Lambda = 1$ ), separating an open universe from a closed one. The red line, nearly along the  $\Lambda = 0$  line, separates a universe that will expand forever (approximately  $\Omega_\Lambda > 0$ ) from one that will eventually recollapse (approximately  $\Omega_\Lambda < 0$ ). And the yellow, nearly vertical line separates a universe with an expansion rate that is currently decelerating from one that is accelerating. The location of three key models are highlighted: standard cold-dark-matter (SCDM) is dominated by matter ( $\Omega_m = 1$ ) and no curvature or cosmological constant; flat ( $\Lambda$ CDM), with  $\Omega_m = 1/3$ ,  $\Omega_\Lambda = 2/3$ , and  $\Omega_k = 0$ ; and Open CDM (OCDM), with  $\Omega_m = 1/3$ ,  $\Omega_\Lambda = 0$  and curvature

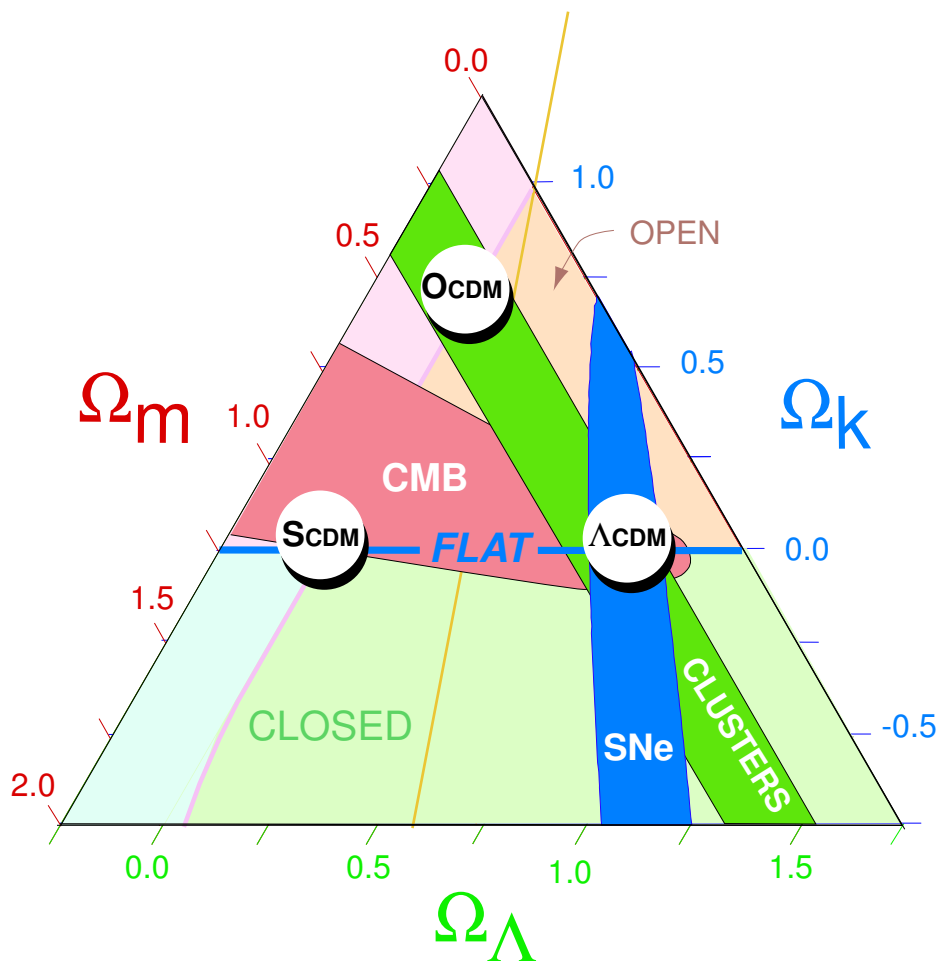


Figure 2: **The Cosmic Triangle Observed** represents current observational constraints. The tightest constraints from measurements at low red shift (clusters, including the mass-to-light method, baryon fraction, and cluster abundance evolution), intermediate red shift (supernovae), and high red shift (CMB) are shown by the three color bands (each representing  $1\text{-}\sigma$  uncertainties). Other tests discussed in the paper are consistent with but less constraining than the constraints illustrated here. The cluster constraints indicate a low-density universe; the supernovae constraints indicate an accelerating universe; and the CMB measurements indicate a flat universe. The three independent bands intersect at a flat model with  $\Omega_m \sim 1/3$  and  $\Omega_\Lambda = 2/3$ ; the model contains a cosmological constant or other dark energy.

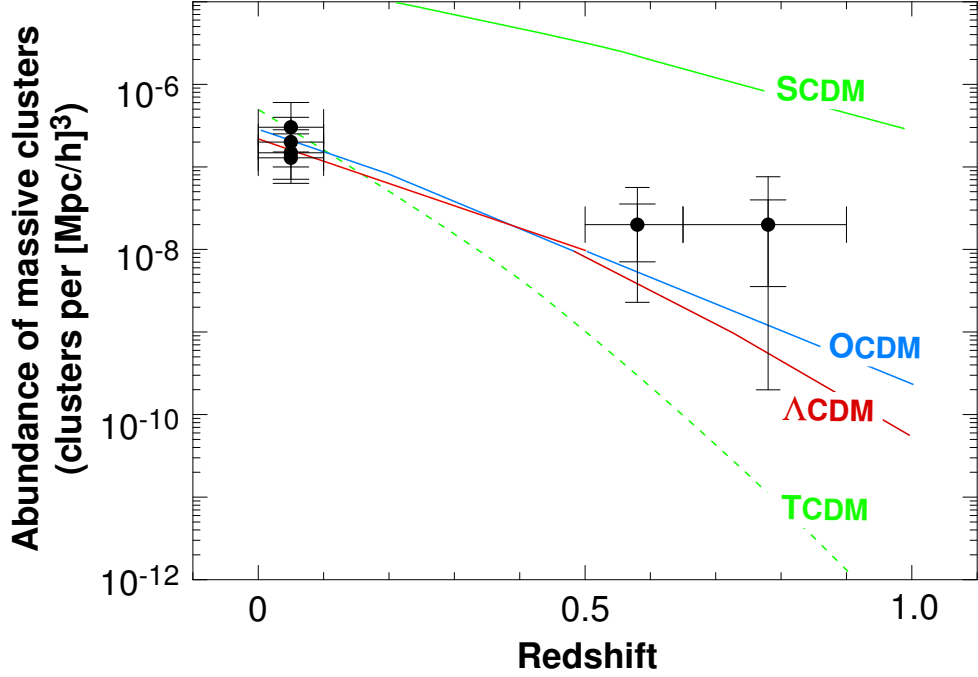


Figure 3: **The evolution of cluster abundance** as a function of red shift is compared with observations [30] for massive clusters (above  $10^{15}$  solar masses within a 2 Mpc radius, assuming  $H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ). Only the  $\Lambda\text{CDM}$  and  $\text{OCDM}$  fit well the observed cluster abundance at  $z \sim 0$ , although the  $\text{TCDM}$  fits much better than the  $\text{SCDM}$  model. See also Fig. 4. All four models are normalized to the cosmic microwave background fluctuations on large scales (see text). The observational data points [30] (with 1- and 2- $\sigma$  error-bars) show only a slow evolution in the cluster abundance, consistent with low  $\Omega_m$  models and inconsistent with  $\Omega_m = 1$ .

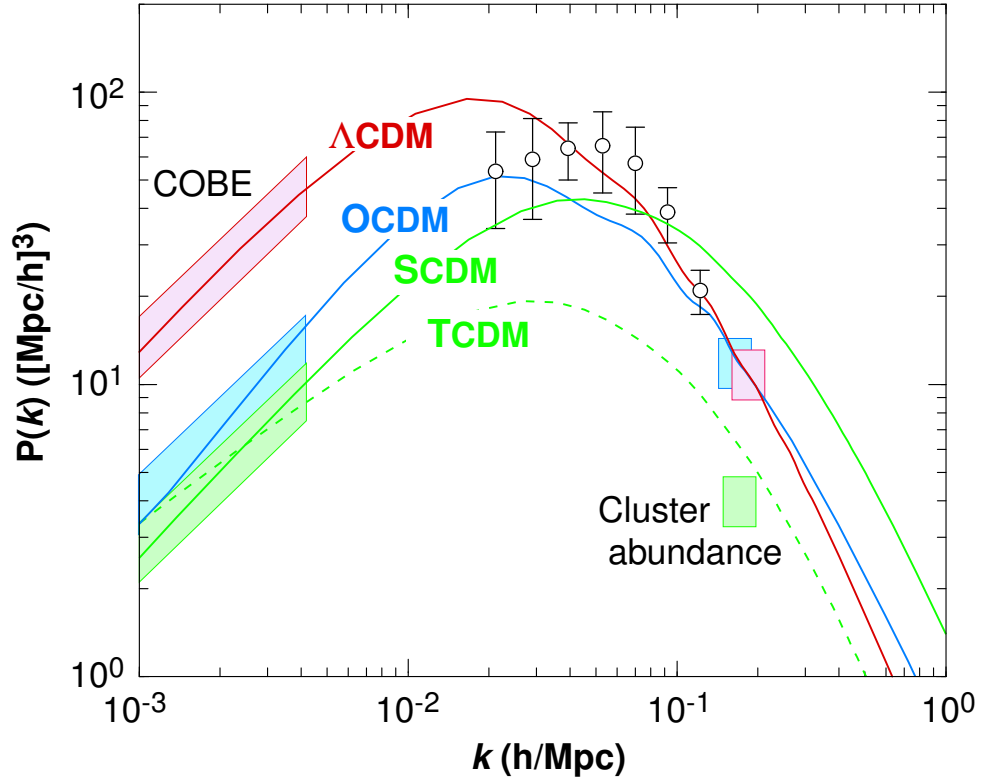


Figure 4: **The mass power spectrum** represents the degree of inhomogeneity in the mass distribution as a function of wavenumber,  $k$ . (The wavenumber is inversely proportional to the length scale; small scales are to the right (large  $k$ 's), and large scales are to the left (small  $k$ 's)). COBE measurements of the CMB anisotropy (boxes on left) and measurements of cluster abundance at  $z \sim 0$  (boxes on right) impose different quantitative constraints for each model; the constraints have been color-coded to indicate the model to which they apply. All curves are normalized to the CMB fluctuations on large scales (*i.e.*, curves are forced to pass through the COBE error boxes on left). Note that the COBE-normalized SCDM model significantly overshoots the cluster constraint (green box on right). The data points with open circles and  $1\sigma$  error bars represent the APM galaxy red shift survey [63]; if one assumes bias, then this set of points can be shifted downwards to match the model, but the shape of the spectrum suggested by the data is unchanged.



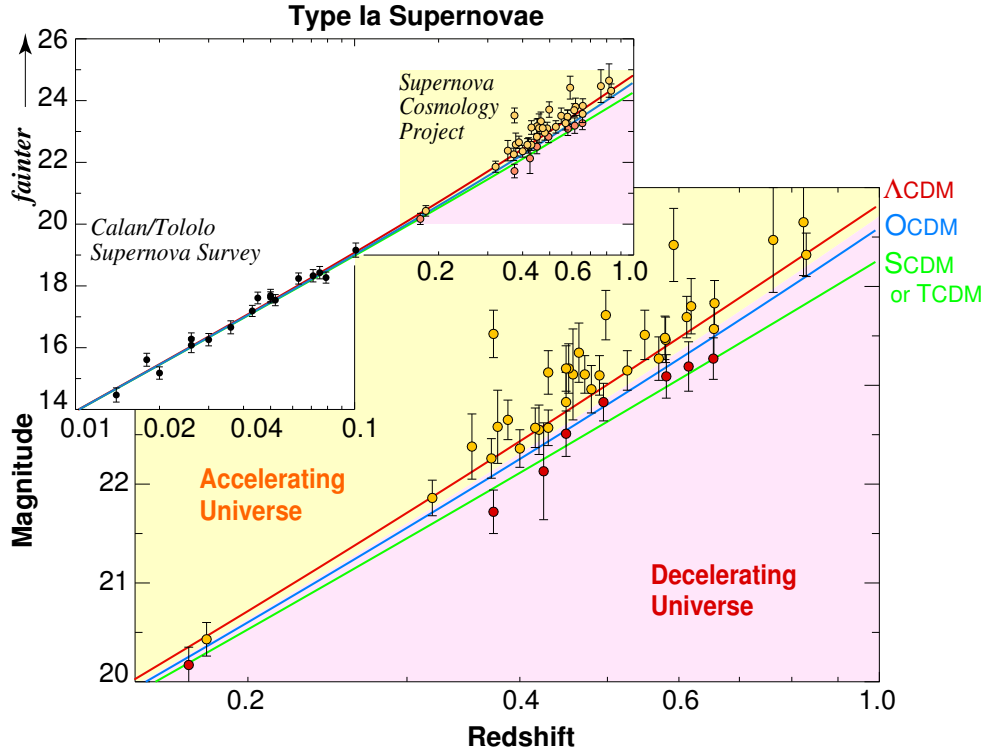


Figure 5: **Supernovae as standard candles**: the relation of observed brightness (in logarithmic units of “magnitude”) vs. red shift for Type Ia supernovae observed at low red shift by the Calan-Tololo Supernova Survey and at high red shift by the Supernova Cosmology Project is presented (with  $1\sigma$  error bars) and compared with model expectations. (Brighter is down and dimmer is up.) (All  $\Omega_m = 1$  flat models yield identical predictions in this method, thus TCDM is identical to SCDM.) The strong gravitational pull exerted by  $\Omega_m = 1$  models (such as TCDM or SCDM), decelerates the expansion rate of the universe and produces an apparent ‘brightening’ of high red shift SNIa, whereas the effect of a cosmological constant accelerating the expansion rate (as in  $\Lambda$ CDM) is seen as a relative ‘dimming’ of the distant SNIa caused by their larger distances. The lower-right plot shows a close-up view of the expected deviations between the three models as a function of red shift. The background color (and shading of the data points) indicates the region for which the universe’s expansion would accelerate (yellow) or decelerate (red) for  $\Omega_m \sim 0.2$ . (Higher values of  $\Omega_m$  would extend the yellow accelerating-universe region farther down on this plot.) Similar results are found by the HZS team [34], as discussed in the text. The results provide evidence for an accelerating expansion rate.

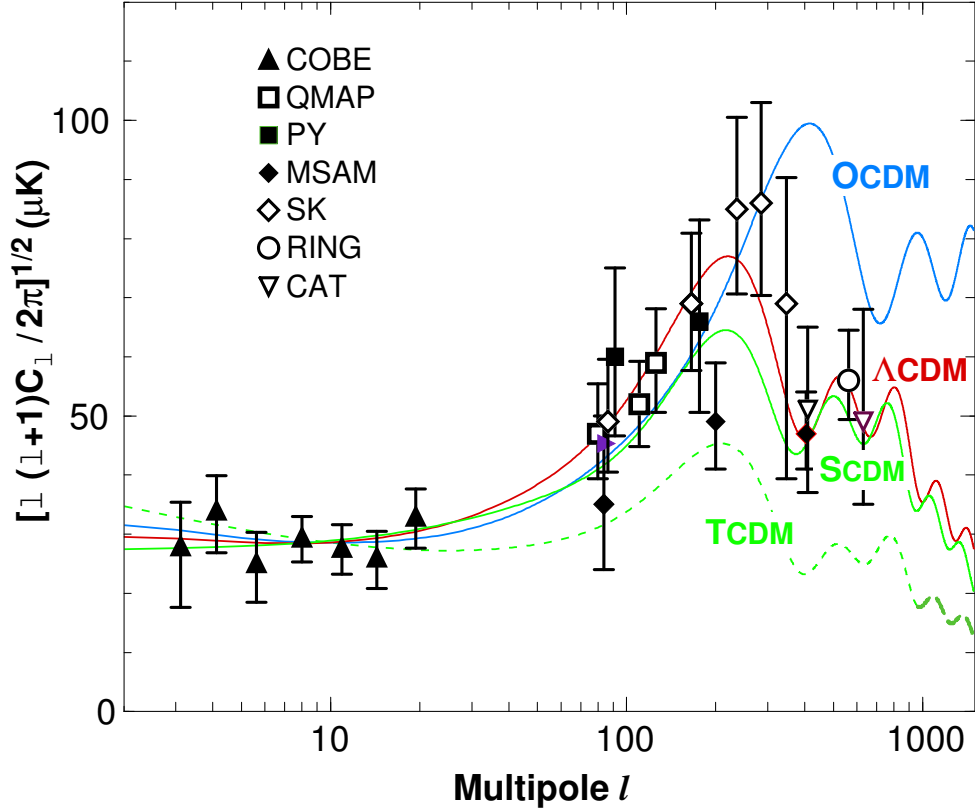


Figure 6: **The cosmic microwave background temperature anisotropy** is presented as a function of angular scale. The multipole  $\ell$  corresponds roughly to an angular scale of  $\pi/\ell$  radians. Flat models ( $\Omega_m + \Omega_\Lambda = 1$ ) produce an acoustic peak at  $\ell \approx 200$  (about one degree on the sky). Open models have a peak that is shifted to smaller scales (larger  $\ell$ 's). (The height of the peak depends on additional parameters, including  $\Omega_m$ ,  $\Omega_\Lambda$ ,  $\Omega_b$ ,  $H_0$ , tilt; here we use the model values from Table I.) The observational data points (with  $1\sigma$  error bars) include the COBE measurements on large scales (small  $\ell$ 's) and other published, multi-frequency ground- and balloon-based observations (see text for references).

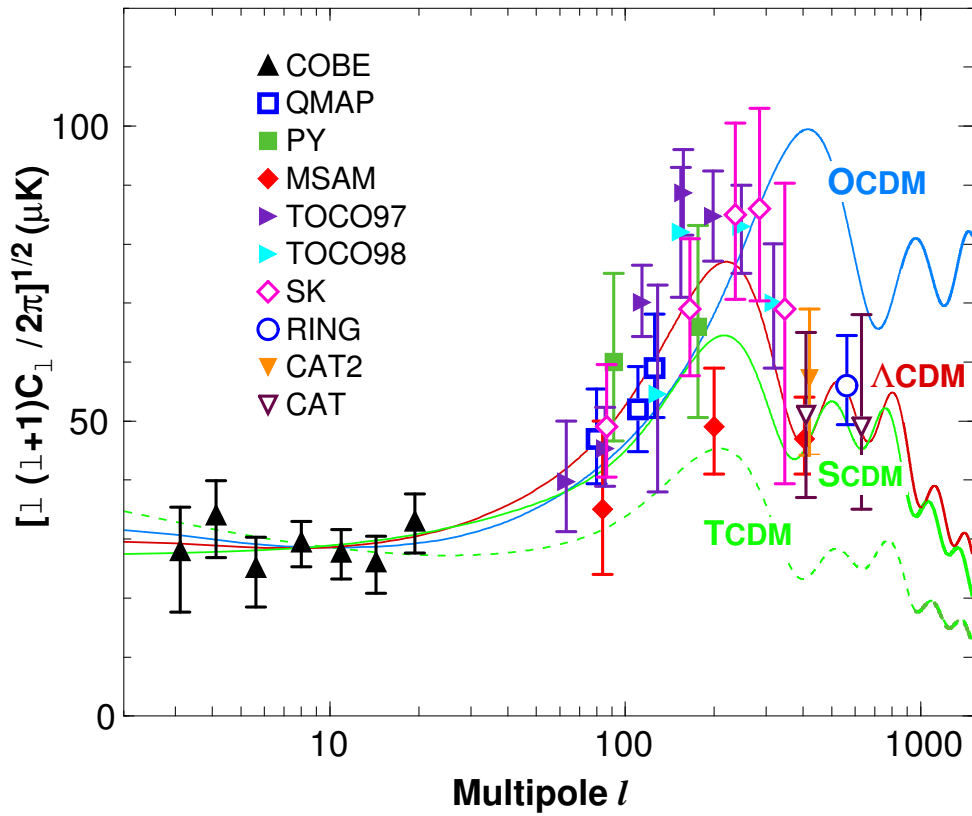


Figure 7: Same as previous figure except with data from recent preprints added.

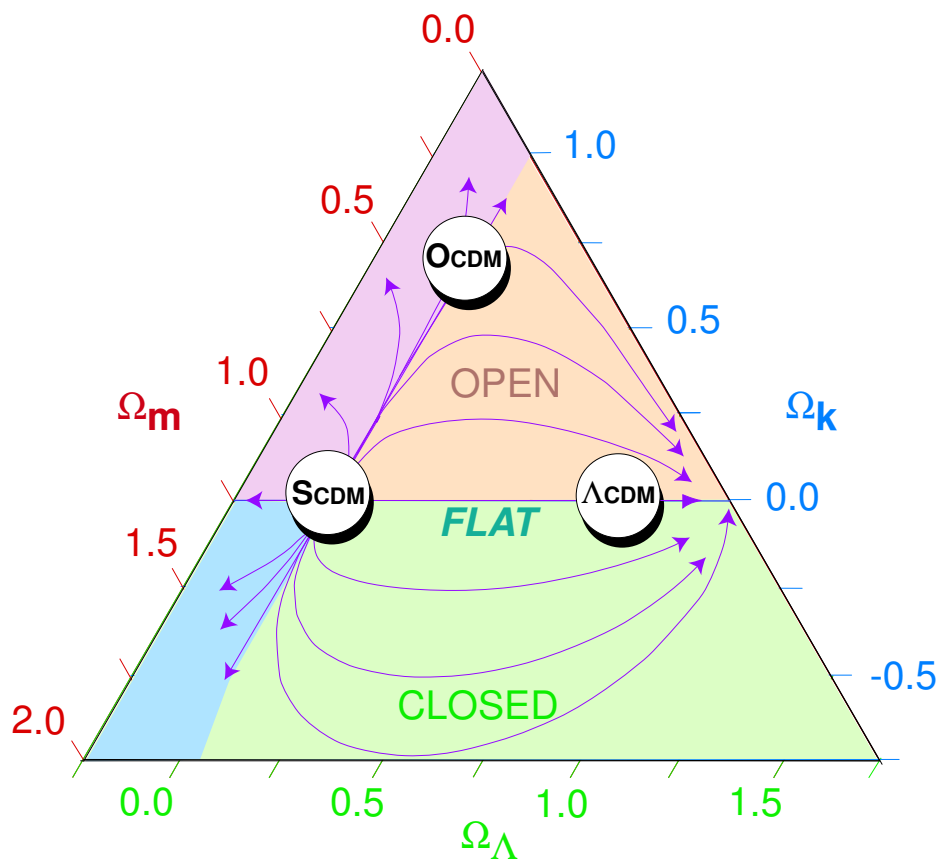


Figure 8: **The past and future of the universe** are represented by various trajectories in the cosmic triangle plot. The trajectories, which originate from near  $\Omega_m = 1$  (an unstable equilibrium point matching the approximate condition of the universe during early structure formation), indicate the path traversed in the triangle plot as the universe evolves. For the current best-fit  $\Lambda$ CDM model, the future represents a flat, accelerating universe that expands forever, ultimately reaching  $\Omega_m \rightarrow 0$  and  $\Omega_\Lambda \rightarrow 1$ .

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