Outflows & Jets: Theory & Observations

Lecture - summer term 2011

Henrik Beuther & Christian Fendt

More Information and the current lecture files:
www.mpi.a.de/homes/beuther/lecture_ss11.html
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Outflows & Jets: Theory & Observations

Lecture plan & schedule

Summer term 2011
Henrik Beuther & Christian Fendt

15.04   Today: Introduction & Overview ("H.B." & C.F.)
29.04   Definitions, parameters, basic observations (H.B.)
06.05   Basic theoretical concepts & models; MHD (C.F.)
13.05   MHD & plasma physics; applications (C.F.)
20.05   Radiation processes (H.B.)
27.05   Observational properties of accretion disks (H.B.)
03.06   Accretion disk theory and jet launching (C.F.)
10.06   Theory of interactions: entrainment, Instabilities, shocks (C.F.)
17.06   Outflow-disk connection, outflow entrainment (H.B.)
24.06   Outflow-ISM interaction, outflow chemistry (H.B.)
01.07   Outflows from massive star-forming regions (H.B.)
08.07   Observations of extragalactic jets (C.F.)
15.07   Some aspects of relativistic jet theory (C.F.)
Outflows & Jets: Theory & Observations

MHD theory

Three approaches to describe an ionized gas

- 1) test particles

  - 2) plasma physics (two fluid components)

    - 3) MHD (one-fluid approach)

see e.g.: http://www.plasma-universe.com/index.php/Plasma-Universe.com
1) ions & electrons

- spiraling along magnetic field, Lorentz force: \( \vec{F} = q \vec{v} \times \vec{B} \)

- gyro radius / Larmor radius: \( r_L \equiv \frac{v}{\Omega_L} \)

- gyro frequency/ cyclotron frequency: \( \Omega_L = \frac{eB}{m_e,i} \)

-> gas is “magnetized” if characteristic length scale \( L \gg r_L \)

-> gas moves on “straight” trajectories if \( L \ll r_L \)

-> magnetization parameter (e, i): \( \delta_{i,e} = \frac{r_L}{L} \)

plasma magnetized if \( \delta_i = \frac{r_{L,i}}{L} \ll 1 \)
Outflows & Jets: Theory & Observations

MHD theory

1) ions & electrons

-> examples:

AGN jet (IC 4296): \[ B \sim 10 \mu G, v \sim c_s \sim 0.005c \rightarrow r_L = 0.03\, cm \]
ISM, protons: \[ B \sim 3 \mu G, 10^{17}\, eV \rightarrow r_L = 30\, pc \]
Galactic field, protons: \[ B \sim \mu G, 10^{19}\, eV \rightarrow r_L = 3\, kpc \]

UHE Cosmic Rays \sim 10^{21}\, eV, \text{ origin unknown}; particles escape if \[ L \ll r_L \]
-> estimate of particle maximum energy in generation process

-> Radiation from gyrating articles:

cyclotron & synchrotron emission, Bremsstrahlung
hot AGN jet plasma -> relativistic thermal motion of particles \[ \gamma < 1000 \]
-> compare to relativistic bulk motion of jet \[ \Gamma < 10 \]

See Larmor radius calculator @ http://pps.coe.kumamoto-u.ac.jp/streaming/PulsedPower/formulary/cal-Lr.html
2) **plasma physics:**

- many particles  ->  statistical theory  ->  collective forces

- **Quasi-neutrality:**

  particle (number) densities [cm\(^{-3}\)]:  \( n_i \sim n_e \sim n \)

- Plasma **kinetic temperature** (in energy units [eV, keV]):  \( k T_{e,i} \equiv \frac{1}{3} m_{e,i} \langle v^2 \rangle \)

  thermal speed:  for  \( T_i = T_e = T \)

  - \( v_{th,e,i} \equiv \sqrt{2 k T / m_{e,i}} \)
2) plasma physics: \(\rightarrow\) many particles \(\rightarrow\) statistical theory \(\rightarrow\) collective forces

\(-\rightarrow\) Plasma frequency:
\(\text{collective dynamic behaviour: charge separation: } \sigma = e n \delta x = -E_x / 4 \pi\)

\(-\rightarrow\) electrostatic oscillation in electric field: \(m \frac{d^2}{dt^2} \delta x = e E_x = -m \omega_p^2 \delta x\)

\(-\rightarrow\) electron plasma frequency:
\(\omega_{p,e}^2 = 4 \pi n_e e^2 / m_e\)
\(\omega_{p,e} [s^{-1}] = 5.64 \times 10^4 n_e^{1/2} [cm^{-3}]\)

\(-\rightarrow\) most fundamental time-scale in plasma physics

\(-\rightarrow\) observable only if 1) oscillation period << life time of system:
\(2)\) external forcing slower than \(\omega_p\)

\(-\rightarrow\) if \(L < v_{th} / \omega_p\) \(-\rightarrow\) plasma behaviour not detected, particles escape

\(-\rightarrow\) critical distance: **Debye length** \(\lambda_D \equiv (kT / m)^{1/2} \omega_p^{-1} \ll L\) for a plasma
2) plasma physics

-> Debye shielding: calculate average Coulomb force by charged particles:
  -> Coulomb potential of test charge $Q$
  -> no plasma: $\Phi = Q/ r$
  -> within plasma: polarization: charge density $\sigma = q (n' - n)$
    (undisturbed and disturbed density of charges $n$ and $n'$)
  -> Poisson equation: $\Delta \Phi = -4\pi \sigma - 4\pi Q \delta(r)$
  -> thermodynamic equilibrium:
    Boltzmann distribution of charged particles: $n' = n \exp(-q \Phi/kT)$
    -> Boltzmann potential $\Phi$ should be local potential (not averaged)
      $\langle \exp(-q \Phi/kT) \rangle \neq \exp(-q \langle \Phi \rangle/kT)$
    -> Taylor expansion (far from charge $Q$):
      $\langle \exp(-q \Phi/kT) \rangle = 1 - \langle q \Phi/kT \rangle$ and $\sigma = -nq^2 \Phi/kT$

see http://farside.ph.utexas.edu/teaching/plasma/lectures/lectures.html
2) **plasma physics**

-> solution of Poisson equation

\[ \frac{d^2}{dr^2}(r \Phi) = \frac{1}{r^2_D} (r \Phi) \]

with boundary conditions

\[ \Phi \rightarrow \frac{Q}{r} \quad \text{for} \quad r \rightarrow 0 \]

\[ \Phi \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty \]

gives

\[ \Phi = \frac{Q}{r} \exp\left(-\frac{r}{\lambda_D}\right) \]

Debye length:

\[ \lambda_D = \left(\frac{kT}{4\pi n q^2}\right)^{1/2} \]

\[ \lambda_D = 743 T^{1/2} n^{-1/2} \text{ cm} \]

\[ \lambda_D / r_{L,e} = 220 B n^{-1/2} \]

-> self-shielding distance, plasma charges “screen out” test charge

-> collective behaviour of particles, works if \( \lambda_D \ll L \)
2) **plasma physics**

-> Debye number, Debye sphere: \( N_D = n \frac{4 \pi}{3} \lambda_D^3 \)

-> for \( N_D >> 1 \): collective behaviour; for \( N_D < 1 \): independent particles

-> plasma parameter: \( \Lambda \equiv 1/N_D \)

-> ionisation degree = relative number of ions and atoms: \( \zeta = n_i/n_a \)

-> depends on ionizing processes

-> thermodynamic equilibrium: temperature-dependent only: \( \frac{n_i}{n_a} \sim g_e \exp\left(-\frac{\Phi_i}{kT}\right) \)

Saha equation: \( \frac{\zeta^2}{1-\zeta} \sim \frac{(kT m_e)^{3/2}}{n h^3} \exp(\Phi_i/kT) \)

-> \( \zeta \sim 0.01\% \) sufficient to behave collectively as plasma

-> mean free path: \( \lambda = \nu_{\text{thermal}} / \nu_{\text{coll}} \) = typical distance between collisions

collision-dominated plasma for \( \lambda \ll L \), typically \( \nu_{\text{coll}} \sim \omega_p (\ln \Lambda / \Lambda) \)

collisionless plasma for \( \lambda > L \), e.g. coupling by magnetic field

-> help establishing Boltzmann distribution

\[
\lambda [cm] = 1.44 \times 10^{13} (\ln \Lambda) n / (kT_e)^2 [eV, cm^{-3}]
\]
2) **plasma physics**

-> parameters of different plasmas
Outflows & Jets: Theory & Observations

MHD theory

2) **plasma physics**

-> summary of parameters of different plasmas

<table>
<thead>
<tr>
<th>Plasma</th>
<th>$n_e$ (m$^{-3}$)</th>
<th>$T$ (K)</th>
<th>$B$ (T)</th>
<th>$\lambda_D$ (m)</th>
<th>$N_D$</th>
<th>$\omega_p$ (s$^{-1}$)</th>
<th>$\nu_{ee}$ (s$^{-1}$)</th>
<th>$\omega_c$ (s$^{-1}$)</th>
<th>$r_L$ (m)</th>
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<td>$10^{-4}$</td>
<td>$10^4$</td>
<td>$10^{10}$</td>
<td>$10^5$</td>
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</table>
Outflows & Jets: Theory & Observations

MHD theory

2) **plasma** physics: fluid model, kinetic theory

-> plasma physics: closure of Maxwell equations
   by expressions for charge density $\rho_c$ and electric current density $j$
   in terms of $\mathbf{E}$ and $\mathbf{B}$
   by microscopic distribution functions for each plasma species

-> define $F_S(r, v, t)$ as **microscopic phase-space density** of plasma species $s$
   near point $(r, v)$ at time $t$.

$F_S$ normalized to particle density in coordinate space: $\int F_s(r, v, t) \, d^3v = n_s(r, t)$,

-> phase space conservation requires:

$$\frac{\partial F_s}{\partial t} + \mathbf{v} \cdot \nabla F_s + \mathbf{a}_s \cdot \nabla_v F_s = 0,$$

while $a_s = \frac{e_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is acceleration of species $s$ by $\mathbf{B}$ and $\mathbf{E}$

-> averaging over ensemble: $F_s = \langle F_S \rangle$ ('a' average plus collision operator):

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{a}_s \cdot \nabla_v f_s = C_s(f).$$

$C_s$ extremely complex

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \bar{a}_s \cdot \nabla_v f_s = 0.$$ for simplicity (Vlasov equation)
2) **plasma** physics: kinetic theory, moments of distribution function

- **density** (1\(^{\text{st}}\) order): 
  \[ n_s(r, t) = \int f_s(r, v, t) \, d^3v, \]

- **flux density** (1\(^{\text{st}}\) order): 
  \[ n_s \, V_s(r, t) = \int v \, f_s(r, v, t) \, d^3v. \]
  \(V_s\) is flow velocity

- **charge density** (1\(^{\text{st}}\) order): 
  \[ \sum_s e_s n_s, \]

- **electric current density** (1\(^{\text{st}}\) order): 
  \[ \sum_s e_s n_s \, V_s. \]

- **stress tensor, momentum flow** (2\(^{\text{nd}}\) order): 
  \[ P_s(r, t) = \int m_s \, v v \, f_s(r, v, t) \, d^3v. \]

- **energy flux density** (3\(^{\text{rd}}\) order): 
  \[ Q_s(r, t) = \int \frac{1}{2} m_s \, v^2 \, v \, f_s(r, v, t) \, d^3v. \]

- **heat flux density** (rest frame): 
  \[ q_s(r, t) = \int \frac{1}{2} m_s \, w_s^2 \, w_s \, f_s(r, v, t) \, d^3v. \]

- **pressure tensor** (rest frame): 
  \[ p_s(r, t) = \int m_s \, w_s w_s \, f_s(r, v, t) \, d^3v, \quad w_s \equiv v - V_s, \]

- **moments in diff. frames**: 
  \[ Q_s = q_s + p_s \cdot V_s + \frac{3}{2} p_s \, V_s + \frac{1}{2} m_s n_s \, V_s^2 \, V_s. \]

- Similar for collision operator ...
2) **plasma** physics: moments of kinetic equation, fluid equations

For each species -> obtain fluid equations by taking moments of ensemble-avaraged kinetic equation:

\[
\frac{\partial f_s}{\partial t} + \nabla \cdot (v f_s) + \nabla_v \cdot (a_s f_s) = C_s(f).
\]

continuity equation:

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s V_s) = 0.
\]

momentum conservation:

\[
\frac{\partial (m_s n_s V_s)}{\partial t} + \nabla \cdot P_s - e_s n_s (E + V_s \times B) = F_s.
\]

energy cons.:

\[
\frac{\partial}{\partial t} \left( \frac{3}{2} p_s + \frac{1}{2} m_s n_s V_s^2 \right) + \nabla \cdot Q_s - e_s n_s E \cdot V_s = W_s + V_s \cdot F_s.
\]

-> fluid equations: re-arrange
  using pressure tensor, heat flux density etc ... -> e.g. \[
\frac{dn_s}{dt} + n_s \nabla \cdot V_s = 0,
\]

-> closure of equations: express viscosity tensor, heat flux, collisional terms in terms of density, velocity, pressure ...

-> hydrodynamic equations (for each species)
Outflows & Jets: Theory & Observations

MHD theory

3) MHD equations

derived from two-fluid plasma equations under certain simplifications:
-> \( m_i >> m_e \); \( v_i \sim v_e \sim v_{\text{thermal}} \); charge neutrality

-> merge two-fluid equations to get one-fluid equation:

example: velocity
\[
\mathbf{v} = \frac{m_i \mathbf{v}_i + m_e \mathbf{v}_e}{m_i + m_e}, \quad \mathbf{j} = -n e \mathbf{U}, \quad \mathbf{U} = \mathbf{v}_e - \mathbf{v}_i
\]

-> \( \mathbf{v}_i \approx \mathbf{v} + O(m_e/m_i), \quad \mathbf{v}_e \approx \mathbf{v} - \frac{\mathbf{j}}{n e} + O\left(\frac{m_e}{m_i}\right). \)

-> from that and \( \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_e) \) and \( \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_i) \) and \( \nabla \cdot \mathbf{j} = 0 \)

follows continuity equation:
\[
\frac{dn}{dt} + n \nabla \cdot \mathbf{v} = 0, \quad \frac{d}{dt} \equiv \partial/\partial t + \mathbf{V} \cdot \nabla
\]

-> similar for equation of motion: add two-fluid equations, total pressure \( p = p_e + p_i \)

\[
m_i n \frac{d\mathbf{v}}{dt} + \nabla p - \mathbf{j} \times \mathbf{B} \approx 0.
\]
3) MHD equations

One-fluid equations + Maxwell equations + Ohm's law + Eq. of state needed for closure

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \left( \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla P - \rho \nabla \Phi + \vec{j} \times \vec{B} + \rho \vec{F}_{rad}$$

$$\partial_t E + \nabla \cdot \left[ \vec{v} \left( E + P + B^2 / 8 \pi \right) - \vec{B} (\vec{v} \cdot \vec{B}) + (\eta \vec{j} \times \vec{B}) \right] = \vec{v} \cdot \left[ -\nabla \Phi + \vec{F}_{rad} \right]$$

$$E = \rho \epsilon + \rho v^2 / 2 + B^2 / 8 \pi, \quad P = \rho \epsilon (\gamma - 1)$$

$$\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B} - \eta \vec{j})$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 4 \pi \vec{j}$$
3) MHD equations, flux freezing

Alfven's theorem (1943):

“In a perfectly conducting fluid, magnetic field lines move with the fluid: field lines are "frozen" into the plasma."

-> motion along magnetic field lines does not change the field, motions transverse to the field carry the field with them

Derivation:

Integrate induction equation \( \frac{\partial B}{\partial t} = \nabla \times (v \times B) \).

with Gauss' theorem \( \int_S \nabla \times A \cdot dS = \int_C A \cdot dl \), (\( S \) closed surface enclosing volume \( V \))

and with Stokes' theorem \( \int_V \nabla \cdot A dV = \int_S A \cdot dS \),

(\( C \) is a closed curve around the open surface \( S \);
\( dS = \hat{n} dS \) with the outward unit normal \( \hat{n} \) )

...... see Appendix
MHD theory

Magnetohydrodynamic waves

-> transport information / energy / momentum / angular momentum

-> impose dynamical time scales

-> linearize MHD equations, using \( Q \rightarrow Q_0 + Q' \)

\( Q_0 \): equilibrium quantity, \( Q' \): perturbed quantity

\[
\frac{\delta \rho'}{\delta t} + \rho_0 \nabla \cdot \vec{v}' = 0 \\
\frac{\delta \vec{B}'}{\delta t} - \nabla \times (\vec{v}' \times \vec{B}_0) = 0 \\
\frac{\delta}{\delta t} \left( \frac{P'}{P_0} - \Gamma \frac{\rho'}{\rho_0} \right) = 0 \\
\rho_0 \frac{\delta \vec{v}'}{\delta t} + \nabla P' - \frac{1}{4 \pi} \left( \nabla \times \vec{B}' \right) \times \vec{B}_0 = 0
\]
MHD theory

Magnetohydrodynamic waves

transport information / energy / momentum / angular momentum

-> look for wave-like solutions of linearized MHD equations, \( Q \sim \exp(\mathbf{k} \cdot \mathbf{r} - \omega t) \):

\[
\begin{align*}
\omega \rho' - \rho_0 \mathbf{k} \cdot \mathbf{v}' &= 0 \\
\omega \left( \frac{P'}{P_0} - \Gamma \frac{\rho'}{\rho_0} \right) &= 0 \\
\omega \mathbf{B}' + \mathbf{k} \times (\mathbf{v}' \times \mathbf{B}_0) &= 0 \\
-\omega \rho_0 \mathbf{v}' + \mathbf{k} P' - \frac{1}{4\pi} (\mathbf{k} \times \mathbf{B}') \times \mathbf{B}_0 &= 0
\end{align*}
\]

-> \( \rho'(\rho_0, \mathbf{V}'), P'(P_0, \mathbf{V}'), B'(B_0, \mathbf{V}') \)

-> substitute into linearized e.o.m.:

\[
\left[ \omega^2 - \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{4\pi \rho_0} \right] \mathbf{v}' = \left( \Gamma \frac{P_0}{\rho_0} + \frac{B_0^2}{4\pi \rho_0} \right) \mathbf{k} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{4\pi \rho_0} \mathbf{B}_0 \left( \mathbf{k} \cdot \mathbf{v}' \right) - \frac{(\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{v}' \cdot \mathbf{B}_0)}{4\pi \rho_0} \mathbf{k}
\]
Magnetohydrodynamic waves

Define e.g. \( \mathbf{B}_0 \parallel \mathbf{e}_z \), wave-vector \( \mathbf{k} \) in x-z plane, \( \theta \) is angle between \( \mathbf{B}_0 \) and \( \mathbf{k} \).

\[
\begin{vmatrix}
\omega^2 - k^2 v_A^2 - k^2 v_S^2 \sin^2 \theta & 0 & -k^2 v_S^2 \sin \theta \cos \theta \\
0 & \omega^2 - k^2 v_A^2 \cos^2 \theta & 0 \\
-k^2 v_S^2 \sin \theta \cos \theta & 0 & \omega^2 - k^2 v_S^2 \cos^2 \theta \\
\end{vmatrix}
\cdot
\begin{pmatrix}
v_x \\
v_y \\
v_z \\
\end{pmatrix}
= 0
\]

Alfven speed \( v_A = \sqrt{\frac{B_0^2}{4 \pi \rho_0}} \) and sound speed \( v_s = \sqrt{\Gamma \frac{P_0}{\rho_0}} \).

Eigen equation solvable if determinant of square matrix vanishes

Dispersion relation:

\[
\left( \omega^2 - k^2 v_A^2 \cos^2 \theta \right) \left[ \omega^4 - \omega^2 k^2 \left( v_A^2 + v_S^2 \right) + k^4 v_A^2 v_S^2 \cos^2 \theta \right] = 0
\]
MHD theory

Magnetohydrodynamic waves

Three independent roots:

1) \( \omega = k v_A \cos \theta \) eigenvector \((0, V_y, 0)\), \( \vec{k} \cdot \vec{v}' = 0 \), \( \vec{v}' \cdot \vec{B}_0 = 0 \)

-> shear Alfven wave; no perturbation of plasma density; motion \_|_\ field

2) \( \omega = k V_+ \),

3) \( \omega = k V_- \),

with \( V_{\pm} = \left[ \frac{1}{2} \left( V_A^2 + V_S^2 \pm \sqrt{(V_A^2 + V_S^2)^2 - 4 V_A^2 V_S^2 \cos^2 \theta} \right) \right]^{1/2} \)

Note that \( V_+ \geq V_- \), eigenvector \((V_x, 0, V_z)\) \( \vec{k} \cdot \vec{v}' \neq 0 \), \( \vec{v}' \cdot \vec{B}_0 \neq 0 \)

-> perturbations in density & pressure, motion || and \_|_\ to magnetic field

-> fast magnetosonic wave (2) and slow magnetosonic wave (3)

For \( V_A \gg V_S \) (strong field), slow wave dispersion reduces,

-> sound wave along magnetic field lines \( \omega \approx k V_S \cos \theta \).
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MHD theory

Magnetohydrodynamic waves

Slow wave

Friedrichs diagram: Phase velocities of 3 MHD waves; low plasma-β with \( V_S < V_A \)

Alfven wave

Fast wave
MHD model of jet formation:

Jets are collimated disk/stellar winds, launched, accelerated, collimated by magnetic forces.
Energy source: gravity, rotation

--> 5 basic questions of jet theory:

- ejection of disk/stellar material into wind?
- **collimation & acceleration** of a disk/stellar wind into a jet
- jet propagation / interaction with ambient medium
- accretion disk structure?
- origin & structure of magnetic field?
3) MHD equations

One-fluid equations + Maxwell equations + Ohm's law + equation of state (needed for closure)

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \]
\[ \rho (\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v}) = -\nabla P - \rho \nabla \Phi + \vec{j} \times \vec{B} + \rho \vec{F}_{rad} \]
\[ \partial_t \vec{E} + \nabla \cdot \left[ \vec{v} \left( \vec{E} + P + \frac{B^2}{8 \pi} \right) - \vec{B} (\vec{v} \cdot \vec{B}) + (\eta \cdot \vec{j}) \times \vec{B} \right] = \vec{v} \cdot \left[ -\nabla \Phi + \vec{F}_{rad} \right] \]

\[ E = \rho \epsilon + \rho \frac{v^2}{2} + \frac{B^2}{8 \pi}, \quad P = \rho \epsilon (\gamma - 1) \]

\[ \partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B} - \eta \vec{j}), \quad \vec{j} / \sigma = \vec{E} + \vec{v} / c \times \vec{B} \]

\[ \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 4 \pi \vec{j} \]
MHD theory

Stationary axisymmetric MHD

Neglect derivatives $t$- and $\phi$-derivatives (Chandrasekhar 1956):

-> Reasonable setup for jet & outflow formation

-> Example: derivation of Ferraro's law of isorotation:

1) use cylindrical coordinates $r$, $\phi$, $z$

2) decompose vectors in
   - poloidal (meridional plane, $r,z$-components)
   - & toroidal components ($\phi$-component): $\vec{B}=\vec{B}_p+\vec{B}_\phi$, $\vec{v}=\vec{v}_p+\vec{v}_\phi$

3) stationary Faraday's law: $\nabla \times \vec{E}=0$ -> potential field $U$ exists: $\vec{E}=\nabla U$

4) since axisymmetry: $E_\phi=0$

5) infinite conductivity $\sigma$: Ohms law: $\vec{E}=-\frac{1}{c}\vec{v} \times \vec{B}$ (\textit{=MHD condition})

6) since $E_\phi=0$ \rightarrow $\vec{v}_p \times \vec{B}_p=0$ or $\vec{v}_p \parallel \vec{B}_p$ \rightarrow $\vec{v}_p = \kappa(r,z)\vec{B}_p$

-> poloidal velocity parallel to poloidal field
Outflows & Jets: Theory & Observations

MHD theory

Stationary axisymmetric MHD

Neglect derivatives $t$- and $\phi$-derivatives (Chandrasekhar 1956)
-> Reasonable setup for jet & outflow formation
-> Example: derivation of Ferraro's law of isorotation:

6) since $E_\phi = 0 \rightarrow \vec{v}_p \times \vec{B}_p = 0$ or $\vec{v}_p \parallel \vec{B}_p \rightarrow \vec{v}_p = \kappa(r, z) \vec{B}_p$

-> poloidal velocity parallel to poloidal field

7) mass conservation & stationarity -> $\nabla(\rho \vec{v}_p) = 0$

8) with $\kappa(r, z)$ -> $0 = \nabla(\rho \kappa \vec{B}_p) = \vec{B}_p \cdot \nabla(\rho \kappa)$

-> $\eta \equiv \rho \kappa$ is conserved along the field lines

9) introduce magnetic flux function:

$$\Psi(r, z) \equiv \frac{1}{2\pi} \iint \vec{B}_p \cdot d\vec{A} = \frac{1}{2\pi} \iint B_z r \, d\phi \, dr$$

$$r \vec{B}_p = -\nabla \Psi \times \vec{e}_\phi$$
Outflows & Jets: Theory & Observations

MHD theory

Stationary axisymmetric MHD

-> Example: derivation of Ferraro's law of isorotation:

9) magnetic flux function:
\[ \Psi (r, z) = \frac{1}{2\pi} \int \int \vec{B}_p \cdot d\vec{A} \]

10) We have further
\[ \vec{v} \times \vec{B} = \vec{v}_\phi \times \vec{B}_p + \vec{v}_p \times \vec{B}_\phi = \frac{1}{r} \left( v_\phi - \frac{n}{\rho} B_\phi \right) \]

11) With MHD condition 5 & Faraday's law:
\[ 0 = \nabla \times \vec{E} = \nabla \Psi \times \nabla \left( \frac{1}{r} \left( v_\phi - \frac{n}{\rho} B_\phi \right) \right) \]

12) Thus the quantity
\[ \Omega_F \equiv \frac{1}{r} \left( v_\phi - \frac{n}{\rho} B_\phi \right) = \Omega_F (\Psi) \]

is conserved along the field line / flux surface

-> Ferraro's law of isorotation, iso-rotation parameter,
“angular velocity of field line”:
\[ \Omega_F = \Omega_F (\Psi) \]
Outflows & Jets: Theory & Observations

MHD theory

Stationary axisymmetric MHD

Conservation laws of stationary MHD:

Mass flow rate per flux surface:
\[ \eta(\Psi) \equiv \rho \frac{v_p}{B_p} \text{sgn}(\vec{v}_p \cdot \vec{B}_p) \]

\[ \dot{M}(\Psi + \Delta \Psi) - \dot{M}(\Psi) = \int_{\Psi}^{\Psi + \Delta \Psi} \rho \vec{v}_p \cdot d\vec{A} = \int_{\Psi}^{\Psi + \Delta \Psi} \eta \vec{B}_p \cdot d\vec{A} \]

Field line iso-rotation:
\[ \Omega_F(\Psi) \equiv \frac{1}{r} \left( v_\phi - \frac{\eta(\Psi)}{\rho} B_\phi \right) \]

Energy conservation:
\[ E(\Psi) \equiv \frac{v^2}{2} - \frac{rB_\phi \Omega_F(\Psi)}{4\pi \eta(\Psi)} + \ldots \quad \text{other energy channels} \]

Angular momentum conservation:
\[ L(\Psi) \equiv r^2 \Omega_F(\Psi) - \frac{rB_\phi}{4\pi \eta(\Psi)} + \ldots \]
Outflows & Jets: Theory & Observations

Stationary MHD - the solar wind

**Application 2: Solar wind**  (Weber & Davis 1967)

-> magnetized solar wind: magnetic wind equation: Alfven Mach number \( M_A = v / v_A \)

-> stellar rotation essential: magnetic field removes angular momentum -> braking

-> radial momentum conservation:

\[
\frac{d}{dr} \left( \frac{1}{2} \nu^2 + \frac{\gamma}{\gamma-1} \frac{P_A}{\rho A} \left( \frac{\rho}{\rho A} \right)^{\gamma-1} - \frac{GM_O}{r} \right) = \frac{v^2}{r} - \frac{1}{8\pi \rho r^2} \frac{dB^2}{dr} \\
\]

-> magnetic wind equation:

\[
\frac{dv}{dr} = \frac{v}{r} \left[ \left( \frac{2\gamma P_A}{\rho A M_A^{2(\gamma-1)}} - \frac{GM_O}{r} \right) \left( M_A^2-1 \right)^3 + \Omega_F^2 r^2 (M_A-1) \left[ (M_A^2+1) M_a - 3 M_A^2 + 1 \right] \right] \\
\]

\[
\left( v^2 - \frac{\gamma P_A}{\rho A M_A^{2(\gamma-1)}} \right) \left( M_A^2-1 \right)^3 - \Omega_F^2 r^2 M_A^2 \left( \frac{r_A^2}{r^2} - 1 \right)^2 \\
\]

-> additional critical points / singularities:

-> **slow magnetosonic point / Alfven point / fast magnetosonic point**

determine critical solution by

\[
v = v_{SM} \quad \text{at} \quad r = r_{SM}, \quad v = v_A \quad \text{at} \quad r = r_A, \quad v = v_{FM} \quad \text{at} \quad r = r_{FM}
\]
Outflows & Jets: Theory & Observations

Stationary MHD - stellar winds

Application 3  Sakurai 1985: axisymmetric structure of MHD jets:

-> dynamics along a given field line by wind equation
-> structure of magnetic field

force-balance across the magnetic field / flux surfaces
-> projection of equation of motion perpendicular to magnetic surfaces \( \Psi(r,z) \):
  - consider \( \phi \)-component of Ampere's law
  - take current density from eq. of motion

-> Grad-Shafranov (GS) equation: (curvature of \( \Psi \))

\[
\left( 1 - \frac{\rho}{\rho_A} \right) v^2 \frac{d\Psi}{ds} = \frac{1}{\rho} \frac{\nabla \Psi}{|\nabla \Psi|} \cdot \nabla \left( \frac{|\nabla \Psi|^2}{8 \pi r^2} + K \rho^\gamma \right) + \frac{\nabla \Psi}{|\nabla \Psi|} \cdot \nabla \Phi_G
\]

\[-\left( L - \frac{r B_\phi}{2 \eta} \right)^2 \frac{1}{r^3} \frac{\partial \Psi}{\partial r} + \frac{1}{\rho r^2} \frac{\nabla \Psi}{|\nabla \Psi|} \cdot \nabla \left( \frac{\pi}{2} (r B_\phi)^2 \right)\]

-> r.h.s.: poloidal magnetic pressure gradient, gas pressure gradient, gravity, centrifugal forces, hoop stress

-> note: GS contains quantities of dynamics calculated from wind equation

-> iterative procedure for solution (Sakurai 1985)
Outflows & Jets: Theory & Observations

Stationary MHD - stellar winds

Axisymmetric structure of stellar MHD outflows:

- dynamics along a given field line by wind equation
- structure of magnetic field

Sakurai 1985:

cross section through flux surfaces
poloidal field lines, critical surfaces
Outflows & Jets: Theory & Observations

Stationary MHD - relativistic wind

Application 4: MHD wind in Kerr metric

-> relativistic motion & metric, relativistically defined velocity: \( u_p \equiv \Gamma v_p / c \)

-> wind equation:

\[
\begin{align*}
\quad u_p^2 + 1 &= -\sigma_m \left( \frac{E}{\mu} \right)^2 \frac{k_0 k_2 + \sigma_m 2 k_2 M_A^2 - k_4 M_A^4}{(k_0 + \sigma_m M_A^2)^2} \\
\quad k_0 &= g_{33} \Omega_F^2 + 2 g_{03} \Omega_F + g_{00} \\
\quad k_2 &= 1 - \Omega_F L / E \\
\quad k_4 &= - \frac{g_{33}^2 + 2 g_{03} L / E + g_{00} L^2 / E^2}{g_{03}^2 - g_{00} g_{33}}
\end{align*}
\]

metric:

\[
\begin{align*}
\quad k_0 &= g_{33} \Omega_F^2 + 2 g_{03} \Omega_F + g_{00} \\
\quad k_2 &= 1 - \Omega_F L / E \\
\quad k_4 &= - \frac{g_{33}^2 + 2 g_{03} L / E + g_{00} L^2 / E^2}{g_{03}^2 - g_{00} g_{33}}
\end{align*}
\]

-> 3 critical points

-> relativistic velocities (Lorentz factor \( \Gamma \), grav. redshift \( \alpha \), frame dragging \( g_{03} \))
Outflows & Jets: Theory & Observations

MHD theory -- simulations

\[ \partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \]
\[ \rho (\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v}) = -\nabla P - \rho \nabla \phi + \vec{j} \times \vec{B} + \rho \vec{F}_{rad} \]
\[ \partial_t \vec{E} + \nabla \cdot [\vec{v} (\vec{E} + P + B^2/8 \pi) - \vec{B} (\vec{v} \cdot \vec{B}) + (\eta \cdot \vec{j}) \times \vec{B}] = \vec{v} \cdot [ -\nabla \phi + \vec{F}_{rad} ] \]

Time-dependent solutions of MHD equations by numerical simulations

-> Numerical MHD codes:
ZEUS, FLASH, PLUTO, NIRVANA ...
applied astrophysical boundary conditions: disk/stellar magnetic field, mass flux ..

-> advantages:
- time-dependent evolution
- no need to search for critical solutions
- 3D structure
- implementation of additional physics “simple” (radiation, viscosity, resistivity ...)

-> difficulties:
- dynamic range: strong density contrast, steep gradients
- computer power limited: grid size, time resolution
Relativistic MHD jets

Relativistic MHD jet formation:

\( \Gamma \sim 1.5 \) (standard setup)
\( \Gamma \sim 6 \) (added Poynting flux by \( B_\phi \))
narrow highly relativistic beam
(< light surface)
mildly relativistic disk wind
(> light surface)

(poloïdal field lines, electric current lines, \( \rho \) in colors)
Outflows & Jets: Theory & Observations

10.10 Introduction & Overview ("H.B." & C.F.)
17.10 Definitions, parameters, basic observations (H.B.)
24.10 Basic theoretical concepts & models I (C.F.): Astrophysical models, MHD
31.10 Basic theoretical concepts & models II (C.F.): MHD, derivations, applications

07.11 Observational properties of accretion disks (H.B.)
14.11 Accretion disk theory and jet launching (C.F.)
21.11 Outflow-disk connection, outflow entrainment (H.B.)
28.11 Outflow-ISM interaction, outflow chemistry (H.B.)
05.12 Theory of outflow interactions; Instabilities (C.F.)
12.12 Outflows from massive star-forming regions (H.B.)
19.12 Radiation processes - 1 (C.F.)
26.12 and 02.01 Christmas and New Year's break
09.01 Radiation processes - 2 (H.B.)
16.01 Observations of AGN jets (C.F.)
23.01 Some aspects of AGN jet theory (C.F.)
30.01 Summary, Outlook, Questions (H.B. & C.F.)
Outflows & Jets: Theory & Observations

Appendix
3) MHD equations, flux freezing

Alfven's theorem (1943):

"In a perfectly conducting fluid, magnetic field lines move with the fluid: field lines are "frozen" into the plasma."

-> motion along magnetic field lines does not change the field, motions transverse to the field carry the field with them

**Derivation:**

Integrate induction equation  \[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \]

with Gauss' theorem \[ \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{l}, \] (S closed surface enclosing volume V)

and with Stokes' theorem \[ \int_V \nabla \cdot \mathbf{A} \, dV = \int_S \mathbf{A} \cdot d\mathbf{S}, \]

(C is a closed curve around the open surface S; \[ d\mathbf{S} = \hat{n} \, dS \] with the outward unit normal \( \hat{n} \))
Outflows & Jets: Theory & Observations

MHD theory

3) MHD equations, flux freezing

(i) Since for all time $\nabla \cdot \mathbf{B} = 0$

$$0 = \int_V \nabla \cdot \mathbf{B} \, dV = \int_S \mathbf{B} \cdot dS, \quad \forall t,$$

(closed surface $S$)

(ii) Time behaviour of magnetic flux $F$ through closed curve $C$, around an open surface $S_1$:

$$\Phi = \int_{S_1} \mathbf{B}(r, t) \cdot dS.$$

$\Phi$ changes in time since $\mathbf{B} = \mathbf{B}(t)$ and since curve $C$ changes in response to plasma motions
3) MHD equations, flux freezing

(iii) curve $C$ moves with the fluid to curve $C'$ within $\delta t$

motion of surface enclosed by $C$ to surface enclosed by $C'$ generates volume $V$ enclosed by surface $S$

(iv) total flux through closed surface $S$ in (iii):

At time $t+\delta t$, when $B(r, t+\delta t)$, we have

$$0 = \int_{\text{closed} S} B(r, t + \delta t) \cdot dS,$$

$$= \int_{\text{top}} B(r, t + \delta t) \cdot dS + \int_{\text{bottom}} B(r, t + \delta t) \cdot dS + \int_{\text{side}} B(r, t + \delta t) \cdot dS,$$

$$= \int_{C'} B(r, t + \delta t) \cdot dS - \int_{C} B(r, t + \delta t) \cdot dS + \int_{\text{side}} B(r, t + \delta t) \cdot dS.$$
3) MHD equations, flux freezing

Consider contribution to total flux from curved side:
A small element of length on the curve $C$ traces out the shaded region.
Then $dS$ is given by the outward normal, $\hat{n}$ times the area of shaded region.
This area is approximately the area of the parallelogram with sides $dl$ and $v\delta t$.
Hence, on the side $dS = dl \times \hat{n} \ \delta t$. Thus,

$$0 = \int_{C'} B(r, t + \delta t) \cdot dS - \int_{C} B(r, t + \delta t) \cdot dS + \int_{C} B(r, t + \delta t) \cdot dl \times v \delta t.$$  

 Flux through curve $C'$ at $t + \delta t$ is equal to flux through curve $C$ minus contribution from sides

$$\int_{C'} B(r, t + \delta t) \cdot dS = \int_{C} B(r, t + \delta t) \cdot dS - \delta t \int_{C} B(r, t + \delta t) \cdot dl \times v,$$
Outflows & Jets: Theory & Observations

MHD theory

3) MHD equations, flux freezing

(vi) Change in flux in time is difference $\Phi (t+\delta t) - \Phi (t)$:

$$
\delta \Phi = \Phi (t + \delta t) \text{ through } C' - \Phi (t) \text{ through } C,
= \int_{C'} B(r, t + \delta t) \cdot dS - \int_C B(r, t) \cdot dS.
$$

with *** (2.41):

$$
\delta \Phi = \int_C \left[ B(r, t + \delta t) - \int_C B(r, t) \right] \cdot dS - \delta t \int_C B(r, t + \delta t) \cdot dl \times v.
$$

small $\delta t \rightarrow$ approximate integrand in surface integral: $B(r, t + \delta t) - \int_C B(r, t) \rightarrow \delta t \partial B / \partial t$

$$
\rightarrow \delta \Phi = \delta t \int_C \frac{\partial B}{\partial t} \cdot dS - \delta t \int_C v \times B \cdot dl. \quad \text{(identity } B \cdot (dl \times v) = v \cdot (B \times dl) = (v \times B) \cdot dl)\n$$

(vii) w/ induction eq.: $\frac{\delta \Phi}{\delta t} = \int_C \nabla \times (v \times B) \cdot dS - \int_C v \times B \cdot dl$, = $\int_C v \times B \cdot dl - \int_C (v \times B) \cdot dl$, on using Stoke's theorem,

$$
= 0.
$$

As $\delta t \rightarrow 0$, $\frac{\delta \Phi}{\delta t} \rightarrow \frac{d \Phi}{dt}$, thus $\Phi$ does not change in time, $\frac{d \Phi}{dt} = \frac{d}{dt} \left\{ \int_C B \cdot dS \right\} = 0$,

where $C$ is any closed contour moving with the fluid.

$\Rightarrow$ Field lines are frozen into the plasma!