Lecture 6

Protoplanetary disks
Part II

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Lecture 6 overview

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1.1 Observational evidence
**Observational evidence**

The observation of young stars shows that they are generally surrounded by disks. Young stars are identified by their Li abundance and by non-black-body emission arising as hot material lands on the star and the IR excess. Classical T Tauri stars are young stars that have shed their envelope, but are still accreting mass from a disc while weak Line T Tauri stars show signs of little accretion.

The disks are not static structures but rather are evolving in the sense that matter is continuously being transported through the disk towards the center where it is finally accreted. The best observational evidence comes from the SED of these young stars.

Another important constraint are the observed accretion rates onto the star:

**Note**
- Mass accretion rates are consistent with the MMSN
- The accretion rates do not allow the accretion of a stellar mass. For the self-similar isothermal collapse we had estimated a constant accretion rate of about $10^{-6}$ solar masses/year.
- There are bursts in the accretion rates: most of the mass could be accreted during such short duration events (FU Orions bursts)
- There is some evidence that the accretion rate may decrease with age.

Muzerolle et al. 2000
1.2 General momentum flux
General momentum transport

Before we start with the mathematical description of the angular momentum transport in the viscous disk, we revisit the general momentum transport equations.

We are interested in
\[
\frac{\partial}{\partial t} (\rho \vec{v}) \quad \text{i.e. for } i=1,2,3 \quad \frac{\partial}{\partial t} (\rho v_i) = \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t} \quad (1)
\]

The continuity equation \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \) is with the Einstein Summenkonvention

\[
\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x_k} \rho v_k \quad (2)
\]

The Euler equation without gravity is
\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p \quad \text{i.e.}
\]

\[
\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad (3)
\]

Combining 2 and 3 into 1 yields

\[
\frac{\partial}{\partial t} (\rho v_i) = -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} + v_i \frac{\partial \rho v_k}{\partial x_k}
\]

\[
= \delta_{i k} \frac{\partial p}{\partial x_k} - \frac{\partial}{\partial x_k} (\rho v_i v_k)
\]
General momentum transport II

Therefore
\[ \frac{\partial}{\partial t} (\rho v_i) = -\frac{\partial}{\partial x_k} \Pi_{ik} \quad (4) \]
\[ \Pi_{ik} = p \delta_{ik} + \rho v_i v_k \]
momentum flux density tensor

\[ \Pi_{ik} \cdot n_k \]
i component of momentum which flows per time and surface unit in direction of the unit vector in direction n.

In the ideal fluid (no friction) momentum is transported by macroscopic fluid motions (second term) and pressure forces (first term). Note: the units of pressure can also be interpreted as a momentum flux:

\[ [\Pi] = \frac{g \text{ cm/s}}{\text{ cm}^2 \text{ s}} = \frac{g}{\text{ cm} \text{ s}^2} \quad [p] = \frac{\text{ dyn}}{\text{ cm}^2} = \frac{g \text{ cm}}{s^2 \text{ cm}^2} = \frac{g}{\text{ cm} \text{ s}^2} \]

Viscous fluid

In a non-ideal fluid, viscosity leads to the additional non-reversible momentum transport from places of high to places of low velocities. We write this by adding a third term

\[ \Pi_{ik} = p \delta_{ik} + \rho v_i v_k - \sigma_{ik} \]
\[ \dot{=} -\sigma'_{ik} + \rho v_i v_k \quad (5) \]
The so called stress tensor is then

\[ \sigma'_{ik} = -p \delta_{ik} + \sigma_{ik} \]
- diag. pressure tensor
- viscous stress tensor ("Reibungstensor")

The combination of 4 and 5 is equivalent to the Navier-Stokes equation.
1.3 Viscous acc. disk theory
Newton's law of friction in shear flow

Remember Newton’s law for the friction between neighboring layers in a laminar shear flow:

\[ F_s = \eta A \frac{\partial u}{\partial y} \]

where
\( \eta \) = dynamic viscosity
\( \frac{du}{dy} \) = velocity gradient
\( A \) = area of the shearing layers
In a (Keplerian) accretion disk, annuli at different radii rotate around the star at different velocities. This causes, for a viscous fluid, friction between them.

The resulting torque between shearing annuli of radius \( r \) of a viscous disk with a kinematic viscosity \( \nu \) rotating in an external, central gravity field is

\[
T = \pi r^2 \nu \Sigma A = 2\pi r^2 \nu \Sigma A \quad \text{[dyn cm]}
\]

This is an equivalent expression as we have seen for Newton’s law of friction. \( \nu = \eta / \rho \), where \( \eta \) is the dynamic viscosity.

Here, \( A \) is the local rate of shearing, which is for the Keplerian case

\[
A = -\frac{1}{2} r \frac{d\Omega}{dr} = \frac{3}{4} \Omega.
\]

The specific angular momentum per mass is \( j = J / m \), or

\[
j = \Omega r^2 = \sqrt{GMr}.
\]
An annulus of mass \( \delta m = 2\pi r \Sigma \delta r \) has its angular momentum increased by the couple from the faster rotating material inside, and decreased by the couple with the material rotating outside. Thus, the net torque is

\[
\frac{d}{dt} j \delta m = T_i - T_o = T(r) - T(r + \delta r) \approx T(r) - \left[ T(r) + \frac{\partial T}{\partial r} \delta r \right] = -\delta r \frac{\partial T}{\partial r}
\]

where \( \frac{d}{dt} = \partial / \partial t + v \nabla \). In an axial symmetric situation and the case that the gravity field is fixed and only a function of radius:

\[
\frac{\partial j}{\partial t} = 0 \quad j = j(r) \quad \text{so} \quad \frac{dj}{dt} = v_r \frac{\partial j}{\partial r} \quad v_r = \text{radial velocity}
\]

Therefore we find the master equation describing the angular momentum transport:

\[
-M \frac{\partial j}{\partial r} = \frac{\partial T}{\partial r} \quad (1)
\]

T=torque  j=specific ang. momentum

Here, \( \dot{M} = 2\pi r \Sigma v_r \) is the mass flux / accretion rate in [g/s] through radius \( r \). Once \( T \) is found, we can find all other quantities of interest as

\[
\dot{M} = -\frac{\partial T}{\partial j} \quad \Sigma = \frac{T}{4\pi r^2 A \nu}
\]
Alternative derivation

An alternative derivation for a non self-gravitating viscous disk can be found in Pringle, 1981, Ann. rev. Astr. Astrophys., 19, 137. The momentum conservation equation for a non self-gravitating viscous disk can be written in components and using the repeated indexes summing rule, as we have seen earlier.

In the case of a viscous accretion disk, it is natural to use the cylindrical coordinates system \((r, \varphi, z)\). Because of differential rotation, there exist a shear since the rotational velocity \(v_\varphi\) varies with \(r\).

For an otherwise axisymmetric disk, we are therefore most interested in the component of the stress tensor that changes the angular momentum namely \(\sigma_{r\varphi}\) (transport of \(\varphi\) angular momentum in radial direction) which we write:

\[
\sigma_{r\varphi} = \eta \left( \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} \right)
\]

Since \(v_\varphi = \Omega r\), we can write:

\[
\sigma_{r\varphi} = \eta r \frac{\partial \Omega}{\partial r}
\]

Notes:
- For solid body rotation, there is no shear stress, as we expect.
- Since \(\eta > 0\) and that for Keplerian disks \(\partial \Omega / \partial r < 0\), the shear stress \(\sigma_{r\varphi} < 0\). Hence, this component of the stress tensor opposes the motion in the \(\varphi\) direction which means that matter is slowing down (loosing angular momentum).
**Alternative derivation II**

Using again the specific angular momentum $j = r^2 \Omega$, we get for the $\varphi$ component of the equation of motion:

$$
\rho \left( \frac{\partial j}{\partial t} + v_r \frac{\partial j}{\partial r} + v_z \frac{\partial j}{\partial z} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \sigma_{r\varphi} \right)
$$

In our situation, we assume as before (no disk self gravity, no/weak explicit time dependence of $j$, cold, thin disk)

$$
r\Omega^2 \approx \frac{GM}{r^2} \ ; \ \frac{\partial j}{\partial t} \ll v_r \frac{\partial j}{\partial r} \ ; \ v_z \frac{\partial j}{\partial z} \ll v_r \frac{\partial j}{\partial r}
$$

Note that we have neglected the radial pressure gradients since, as we have seen earlier, the disks are nearly Keplerian. With these assumptions, we can integrate the equation above over the entire $z$ coordinate:

$$
\int_{-\infty}^{\infty} \rho v_r \frac{\partial j}{\partial r} \, dz = \int_{-\infty}^{\infty} \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \sigma_{r\varphi} \right) \, dz
$$

Using again $\dot{M} = -2\pi rv_r \Sigma$ and the torque $T = 2\pi r \int_{-\infty}^{\infty} r \sigma_{r\varphi} \, dz$ we again find:

$$
-\dot{M} \frac{\partial j}{\partial r} = \frac{\partial T}{\partial r} \quad \text{T=torque} \quad j=\text{specific ang. momentum}
$$
Master equation III

The signification of this formula can be made graphically clear as follows:

The difference in viscous torque $T$ and $T+dT$ exerted across circles of radii $r$ and $r+dr$ transports angular momentum outward and mass inwards.

With the kinematic viscosity as $v=\eta/\rho$, we have: $\sigma_{r\varphi} = \rho vr \frac{\partial \Omega}{\partial r}$ and for the torque $T$ we find:

$$T = 2\pi r \int_{-\infty}^{\infty} r \left( \rho vr \frac{\partial \Omega}{\partial r} \right) dz = 2\pi r^3 v \frac{\partial \Omega}{\partial r} \Sigma$$

where we have used $\Sigma = \int_{-\infty}^{\infty} \rho dz$ which is the same result for $T$ as at the beginning of the chapter.

In addition to the momentum equation, we must also satisfy the mass conservation equation:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rv_r \Sigma) = \frac{\partial \Sigma}{\partial t} - \frac{1}{2\pi r} \frac{\partial \dot{M}}{\partial r} = 0 \quad (2)$$

From eq. (1) we know

$$\dot{M} = - \left( \frac{\partial j}{\partial r} \right)^{-1} \frac{\partial T}{\partial r}$$
**Master equation IV**

Plugging this into (2), and using the expression for $T$, we have now a closed equation describing the time evolution of the surface density:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[ \left( \frac{\partial j}{\partial r} \right)^{-1} \frac{\partial}{\partial r} \left( r^3 \nu \frac{\partial \Omega}{\partial r} \Sigma \right) \right] = 0$$

For the (quasi) Keplerian disk we can work a little bit more using $\Omega = \Omega_0 r^{-3/2}$

$$\frac{\partial j}{\partial r} = \frac{\partial}{\partial r} r^2 \Omega = \frac{1}{2} \Omega_0 r^{-3/2} \quad \frac{\partial \Omega}{\partial r} = -\frac{3}{2} \Omega_0 r^{-5/2}$$

Putting that back into the equation above we finally find:

$$\frac{\partial \Sigma}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left[ 3 r^{1/2} \frac{\partial}{\partial r} \left( r^{1/2} \nu \Sigma \right) \right] = 0$$

which is the form usually used to express the evolution of the gas surface density of the disk. Note that the viscosity is in general a function of $r$, too. Therefore, this is a nonlinear equations.

The result here is a general one. It does not only apply to a protoplanetary disk only, but for example also to disks around a neutron star or a black hole.
1.4 Viscosity
The master equation is
\[
\frac{\partial \Sigma}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left[ 3r^{1/2} \frac{\partial}{\partial r} \left( r^{1/2} \nu \Sigma \right) \right] = 0
\]

We know that the combination of the continuity equation, and Fick’s law \( \rho \vec{v} = -D \nabla \rho \) gives the standard diffusion equation:
\[
\frac{\partial \rho}{\partial t} - D \Delta \rho = 0
\]

Interestingly, the master equation shows that the surface density obeys a diffusion equation (one derivative in time and two derivatives in space) with the diffusion coefficient given in order of magnitude by the kinematic viscosity.

**Viscous timescale and abnormal viscosity**

The timescale for evolution is therefore given by the viscous timescale which we estimate by:

\[
\tau = \frac{r^2}{\nu}
\]
**What viscosity?**

To the viscous timescale of the disk, we use characteristic values for nebular disks:

\[ r \approx 10^{14} \text{ cm}, \quad \nu_{\text{therm}} \approx 10^5 \text{ cm/s}, \quad \lambda \approx 10 \text{ cm}, \quad \nu \approx \lambda \cdot \nu_{\text{therm}} = 10^6 \text{ cm}^2/\text{s} \]

\[ \tau \approx 10^{22} \text{ s} \approx 3 \times 10^{14} \text{ years} \]

This timescale is 7-8 orders of magnitude longer than the lifetime assigned to these objects, and much more than the age of the universe... If viscosity is to play a role in the evolution of these disks, there must be a source of abnormal viscosity as molecular viscosity cannot be of importance!

Many possible explanation have been proposed to account for the extra-viscosity but one can essentially group them in two main categories depending on whether or not magnetic fields play an important role. The issue is still highly controversial as good arguments can be found on both sides. (Balbus & Hawley, 2000, in From Dust to Terrestrial Planets, Eds. W. Benz, R. Kallenbach & G.W. Lugmair, Kluwer Academic Publisher, Dordrecht).

They however agree on the fact that turbulence must play a certain role in causing the abnormal viscosity.
**Turbulent viscosity**

The general idea is the following: If turbulence exists, the fluctuating velocity field $\delta u$ will give rise to a turbulent shear stress which will have the form

$$\sigma_{r\phi} = -\rho \langle \delta u_r \cdot \delta u_\phi \rangle$$

where the square brackets indicate an average over a suitable ensemble of turbulent fluctuations. If this correlation exists (non-zero), than we can estimate its magnitude simply by stating that each fluctuating component must be of order a fraction of the thermal velocity which is itself roughly equal to sound speed. We therefore write:

$$\sigma_{r\phi} = -\rho \alpha c^2 = -\alpha p$$

Note that in the case of a fluctuating magnetic field $\delta B$ the associated turbulent stress can be written as

$$\sigma_{r\phi} = \frac{1}{4\pi} \langle \delta B_r \cdot \delta B_\phi \rangle \approx -\alpha \rho c^2$$

where we have associated the fluctuating magnetic pressure to a fraction $-\alpha$ of the hydrodynamic pressure.
\( \alpha \text{ viscosity} \)

From before we know

\[ \sigma_{r\varphi} = \rho \nu r \frac{\partial \Omega}{\partial r} \]

therefore we can write

\[ -\alpha \rho c^2 = \nu \rho r \frac{\partial \Omega}{\partial r} \rightarrow \nu = -\frac{\alpha c^2}{r \frac{\partial \Omega}{\partial r}} \]

We can evaluate this expression recalling that for a Keplerian disk:

\[ \frac{\partial \Omega}{\partial r} = -\frac{3 \Omega}{2 r} \]

\[ \nu = \frac{\alpha c^2}{\Omega} = \alpha c H = \alpha \left( \frac{H}{r} \right)^2 \Omega r^2 \]

We have used \( c = H \Omega \) and absorbed the numerical constant into \( \alpha \).

This \( \alpha \) prescription has the virtue to satisfy dimensional considerations and to be extremely simply compared to any other model. In lack of a better model for turbulence, this so-called alpha-model has received considerable attention and is quite often used in the literature.
As for the numerical value of $\alpha$ there is little known. Often, $\alpha$ is taken to be a constant (does not depend on radius) with a value $-4 < \log \alpha < -2$. Note that assuming a constant value for $\alpha$ doesn't assume a constant value for the viscosity as we can see from the last equation. If $H/r$ is follows a powerlaw (as we saw for the MMSN), then the viscosity will also follow a powerlaw as a function of radius.

This last equation, using $\alpha=10^{-3}$, $r=1\text{AU}$ and $(H/r)=0.1$, yields $\nu=1.5 \times 10^{14}$ cm$^2$/s.

This viscosity is eight orders of magnitude larger than the molecular viscosity calculated before. The corresponding viscous disk evolution timescale is therefore eight orders of magnitude shorter which means it is of order Myrs. This brings it in good agreement with the observed disk lifetimes (mean lifetime $\sim$3 Myrs).
2. Turbulence, MRI, baroclinic instability (by PD Dr. Klahr)