Practical Numerical Training UKNum

Interpolation, Extrapolation, Splines

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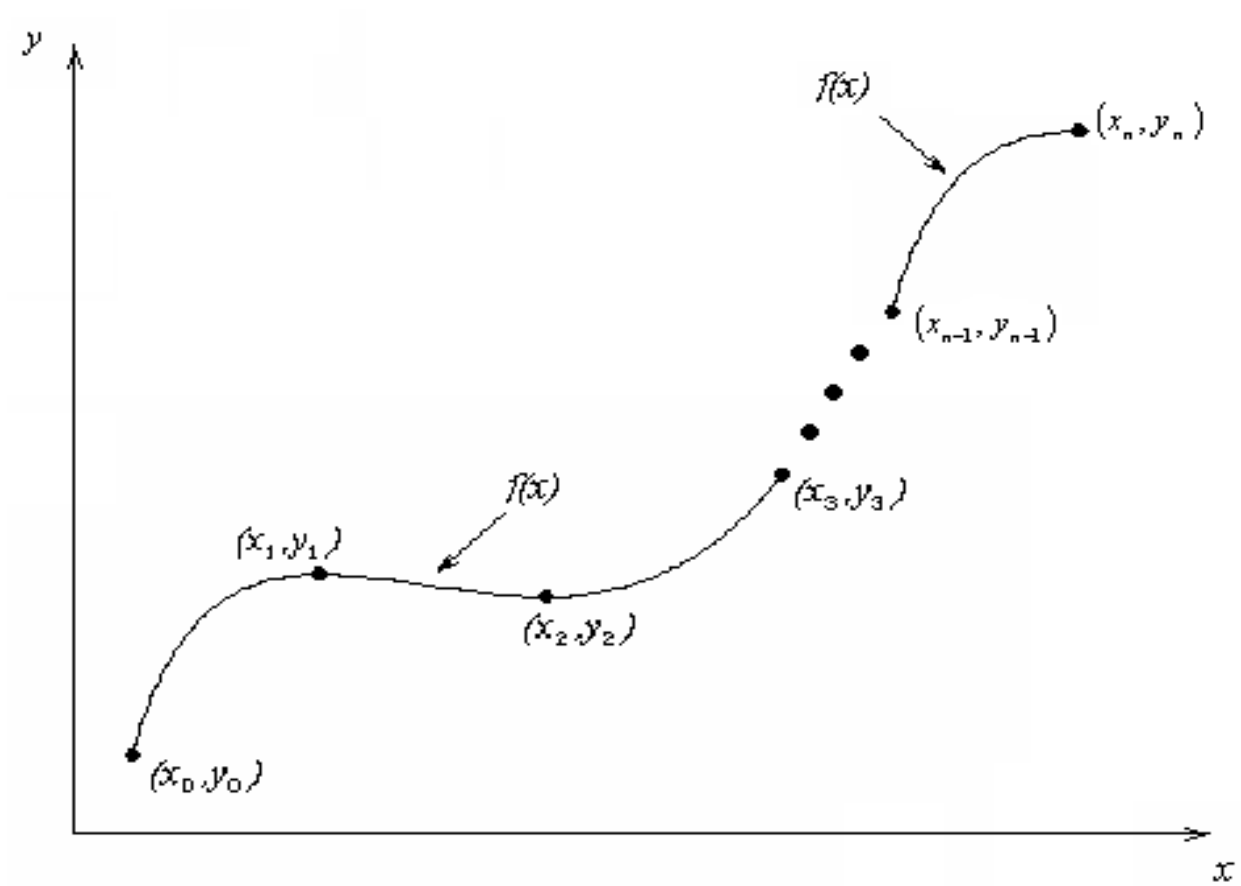
Schedule:
1) Introduction
2) Direct Method
3) Divided Differences Method
4) Splines

1 Intro

Task:

- Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n) discrete data sets
  - E.g. measurements, numerical Results
- Assume ordered values $x_0 < x_1 \ldots < x_N$
- Asking for $y=f(x)$ for an arbitrary value x
  - If $x_0 \leq x \leq x_N$: Interpolation
  - Else: x outside interval: Extrapolation (!)



Principle

- Basic requirement for interpolant $f(x)$
  - For $\forall\ x_i$ it must be $f(x_i)=y_i$

- Different classes of Interpolants
  - Polynomials
  - Rational Functions
  - Trigonometric Functions
  - ...

- Most common interpolants: Polynomials. Because easy to
  - Evaluate
  - Differentiate
  - Integrate

Polynomial interpolation

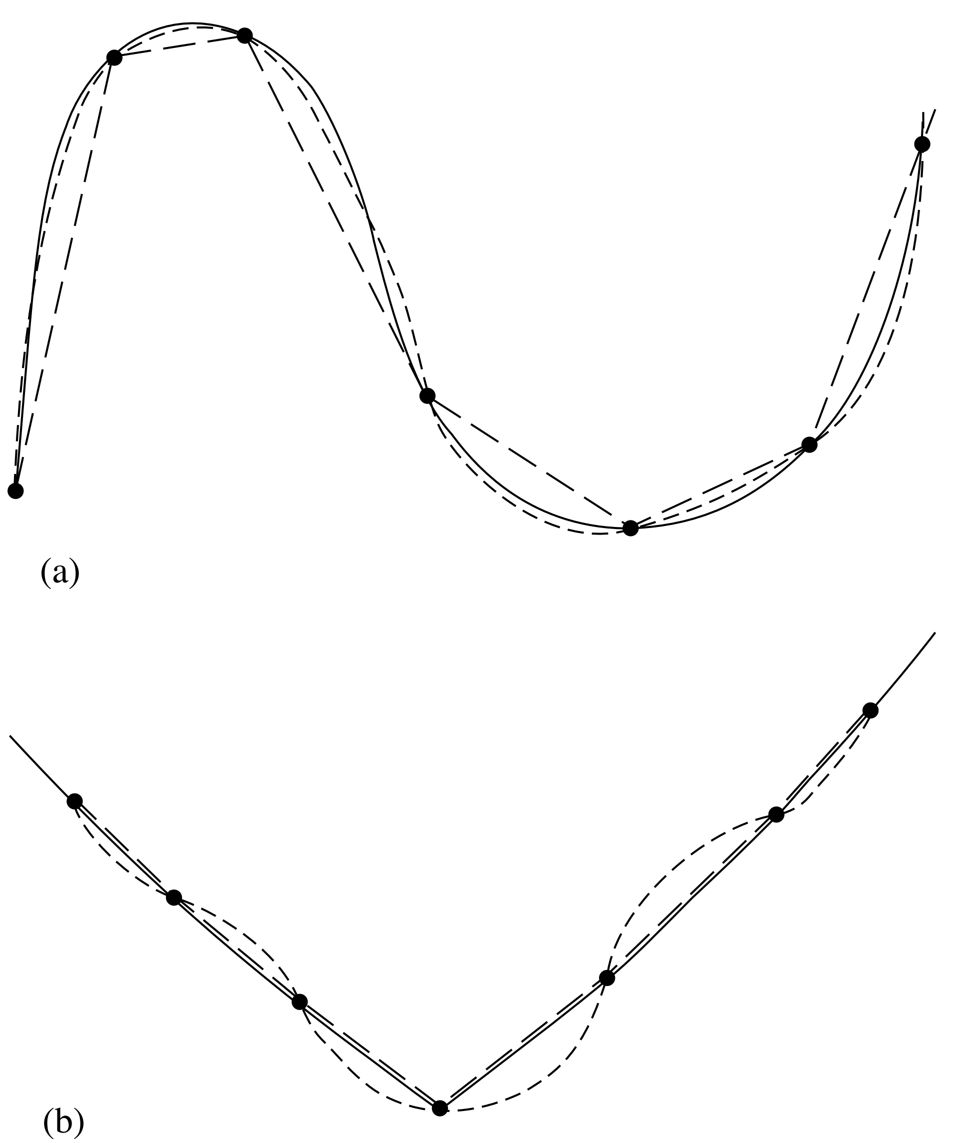
- For n+1 pairwise different data points there is exactly one Interpolation polynomial of n-th order, to fulfil $\forall\ x_i\ f(x_i)=y_i$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

- 0th-order: constant
- 1st-order: linear interpolation
- 2nd-order: quadratic interpolation
- 3rd-order: cubic interpolation

Suggested grade?

- More orders (higher grade) mean more oscillations.



- 2nd. 3rd. and 4th order recommended. Higher orders dangerous!
- Only use points x_i in the direct neighbourhood of x. Piecewise Interpolation (see also Splines).

Warning

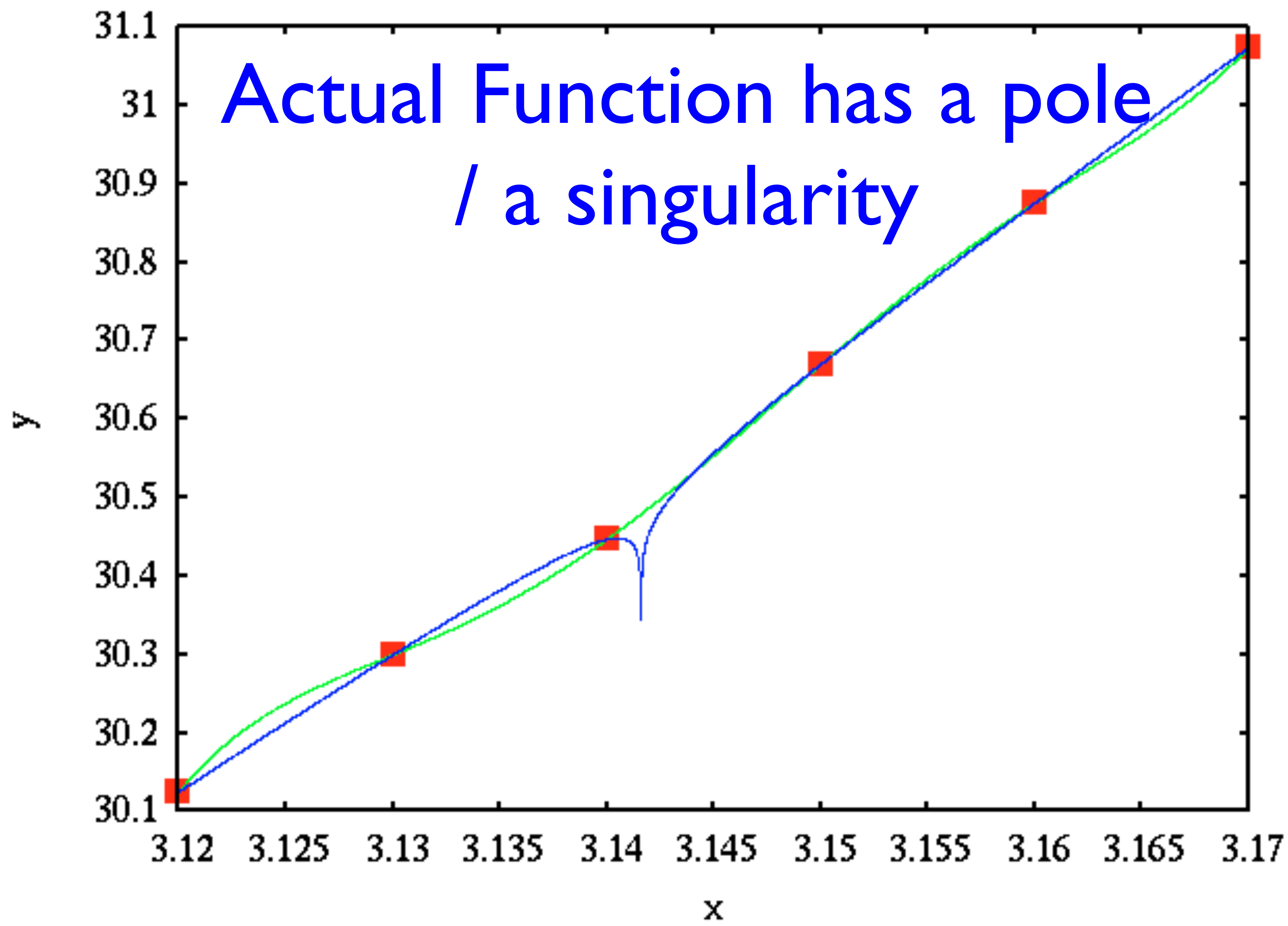
- One generally assumes that the values between tabulated data is smooth.
- But sometimes we do not know…
- If the data is not smooth, interpolated values can strongly deviate:
- Example:

$$f(x) = 3x^2 + \frac{1}{\pi^4} \ln\left[(\pi - x)^2\right] + 1$$

Warning II

- x_i=3.12, 3.13, 3.14, 3.15, 3.16, 3.17

$$f(x) = 3x^2 + \frac{1}{\pi^4}\ln\left[(\pi - x)^2\right] + 1$$



2 Direct Methods

Ansatz

Given 'n+1' data points (x_0,y_0), (x_1,y_1),............. (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n .$$

where a_0, a_1,.................. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.
- Lineare System:

$$\begin{bmatrix} x_0^n & x_0^{n-1} & x_0^{n-2} & \ldots & x_0 & 1 \\ x_1^n & x_1^{n-1} & x_1^{n-2} & \ldots & x_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_n^n & x_n^{n-1} & x_n^{n-2} & \ldots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

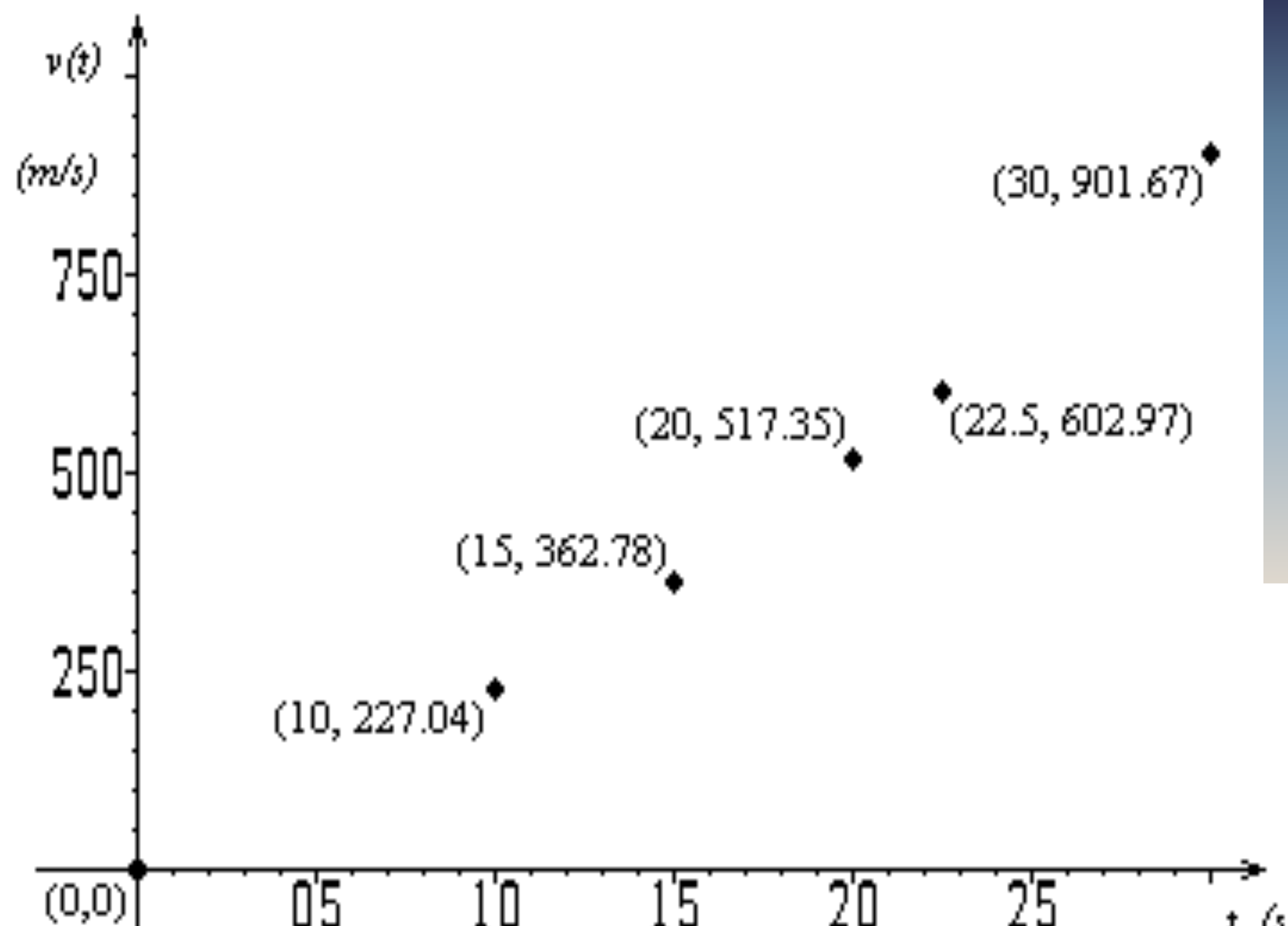
Polynomial interpolation

- So called Wandermonde Matrix.
- Solved via Gaussian Elimination
- Has a solution, but numerically expensive: scales as N^3 and has relative large errors in determining the values for a_i.

Beispiel

- The upward velocity of a rocket is given as a function of time in Table 1.
- Find the velocity at t=16 seconds using the direct method for linear interpolation.

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

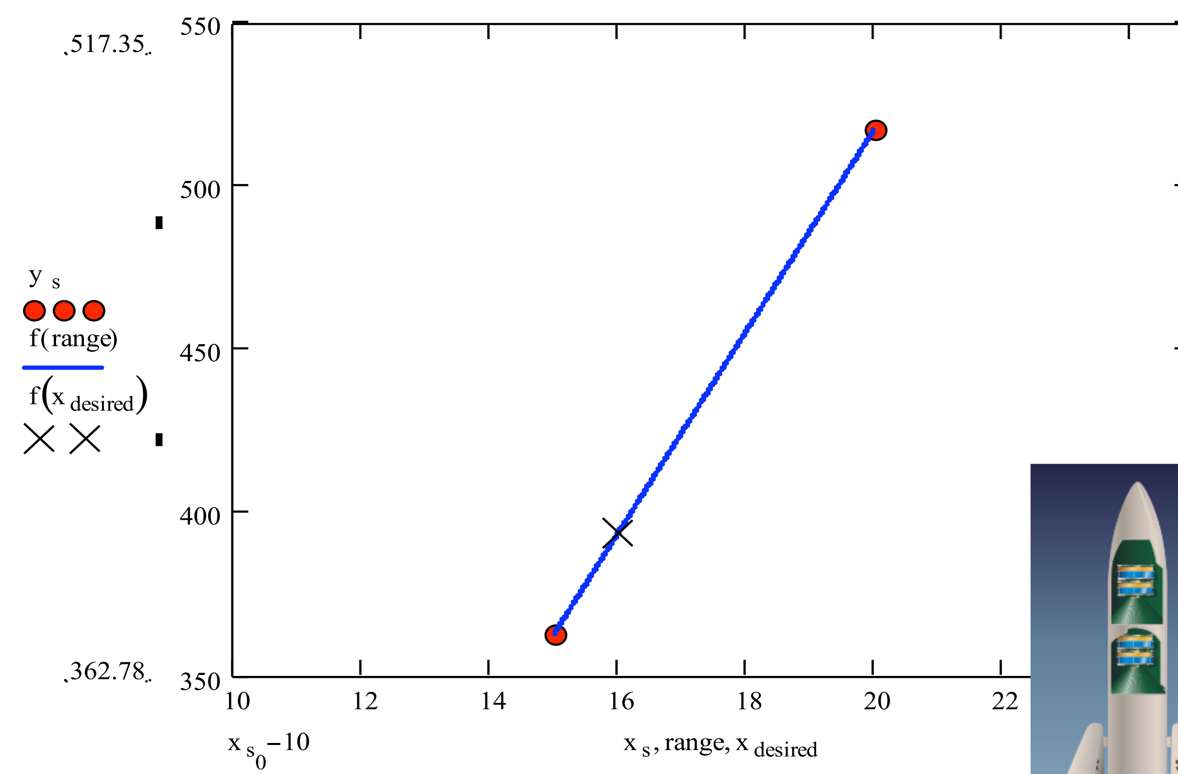


Direct Method, linear

- Ansatz: $$v(t) = a_0 + a_1 t$$

  - Bracketing values:

$$v(15) = a_0 + a_1(15) = 362.78$$
$$v(20) = a_0 + a_1(20) = 517.35$$



- You find

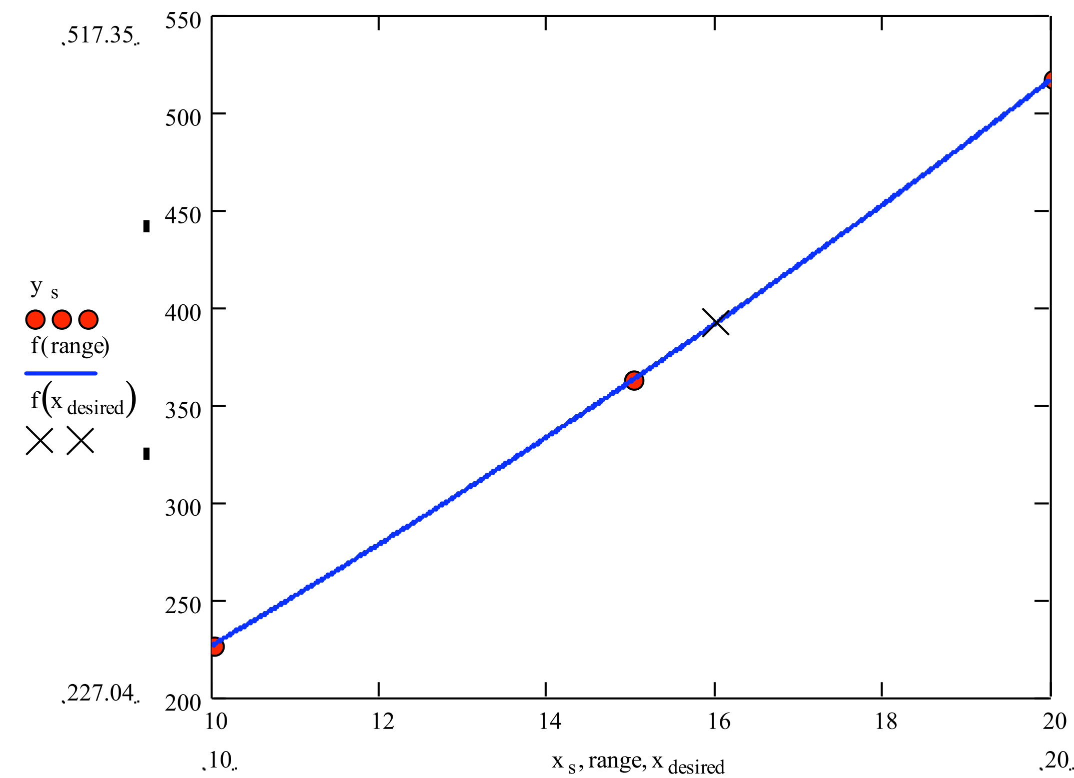
$$a_0 = -100.91 \qquad a_1 = 30.913$$

- Thus

$$v(t) = -100.91 + 30.913t, \;\; 15 \le t \le 20.$$
$$v(16) = -100.91 + 30.913(16) = 393.7\,m/s$$

Directe Method, quadratic

- Ansatz: $v(t) = a_0 + a_1 t + a_2 t^2$
- Bracketing values: (*which ones?*)



$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

- Find:

$$a_0 = 12.001 \quad a_1 = 17.740 \quad a_2 = 0.37637$$

- Thus

$$v(t) = 12.001 + 17.740t + 0.37637t^2, \quad 10 \le t \le 20$$

$$v(16) = 12.001 + 17.740(16) + 0.37637(16)^2$$

$$= 392.19 m/s$$

Direct Method, quadratic II

- Difference between higher and Lower order used for estimate error.
- Error:

$$\left|\in_a\right| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

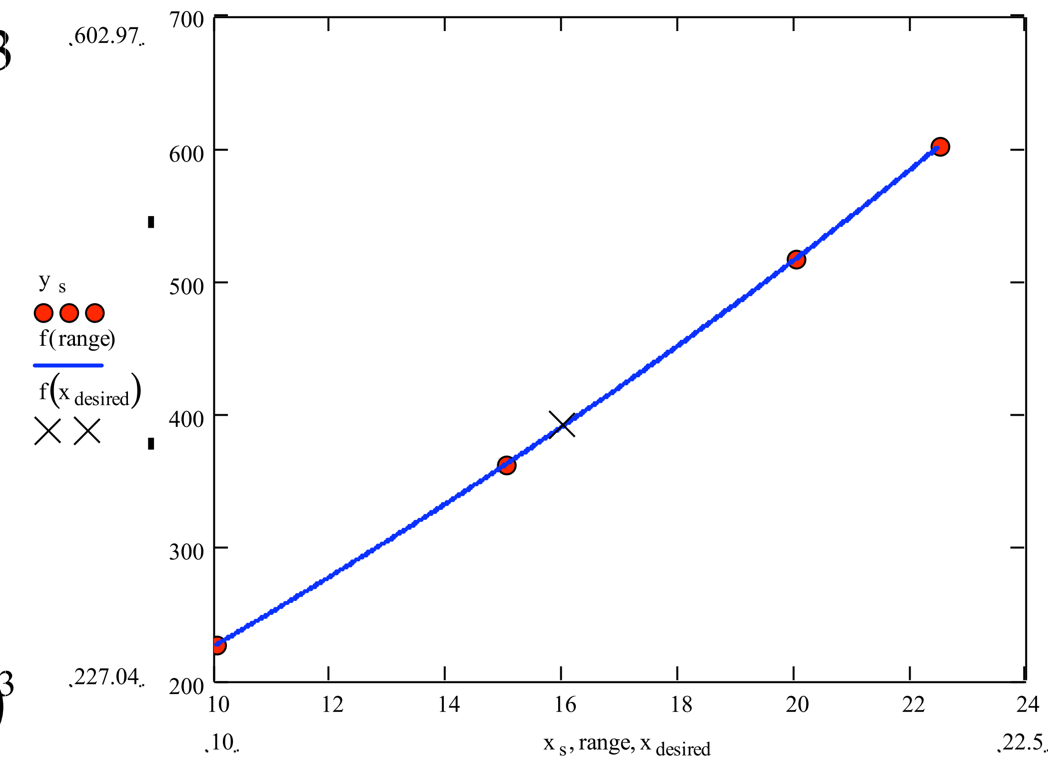
$$= 0.38502\%$$

- Quadratic Contribution is small.

Direct Method, cubic

- Ansatz: $v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
- Bracketing values

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$
$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$
$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$
$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$



- Man findet

$$a_0 = -4.3810 \quad a_1 = 21.289 \quad a_2 = 0.13065 \quad a_3 = 0.0054606$$

- Somit

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$
$$v(16) = -4.3810 + 21.289(16) + 0.13064(16)^2 + 0.0054606(16)^3$$
$$= 392.06 m/s$$

Direct Method, cubic II

- Error with respect to quadratic solution:

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$

$$= 0.033427\%$$

- Cubic contribution very small

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	----------	0.38502 %	0.033427 %

- Increasing order -> lower error -> Convergence

Covered Distance? Integral

What distance was covered between t = 10 and t = 22.5 sec?

- Polynome can be easily integrated

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$\approx \int_{11}^{16} \left(-4.3810 + 21.289t + 0.13065t^2 + 0.0054606t^3\right)dt$$

$$= \left[-4.3810t + 21.289\frac{t^2}{2} + 0.13065\frac{t^3}{3} + 0.0054606\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605\ m$$

Acceleration

What was the acceleration at t = 16 sec?

- Polynomials can be easily differentiated:

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(-4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3\right)$$

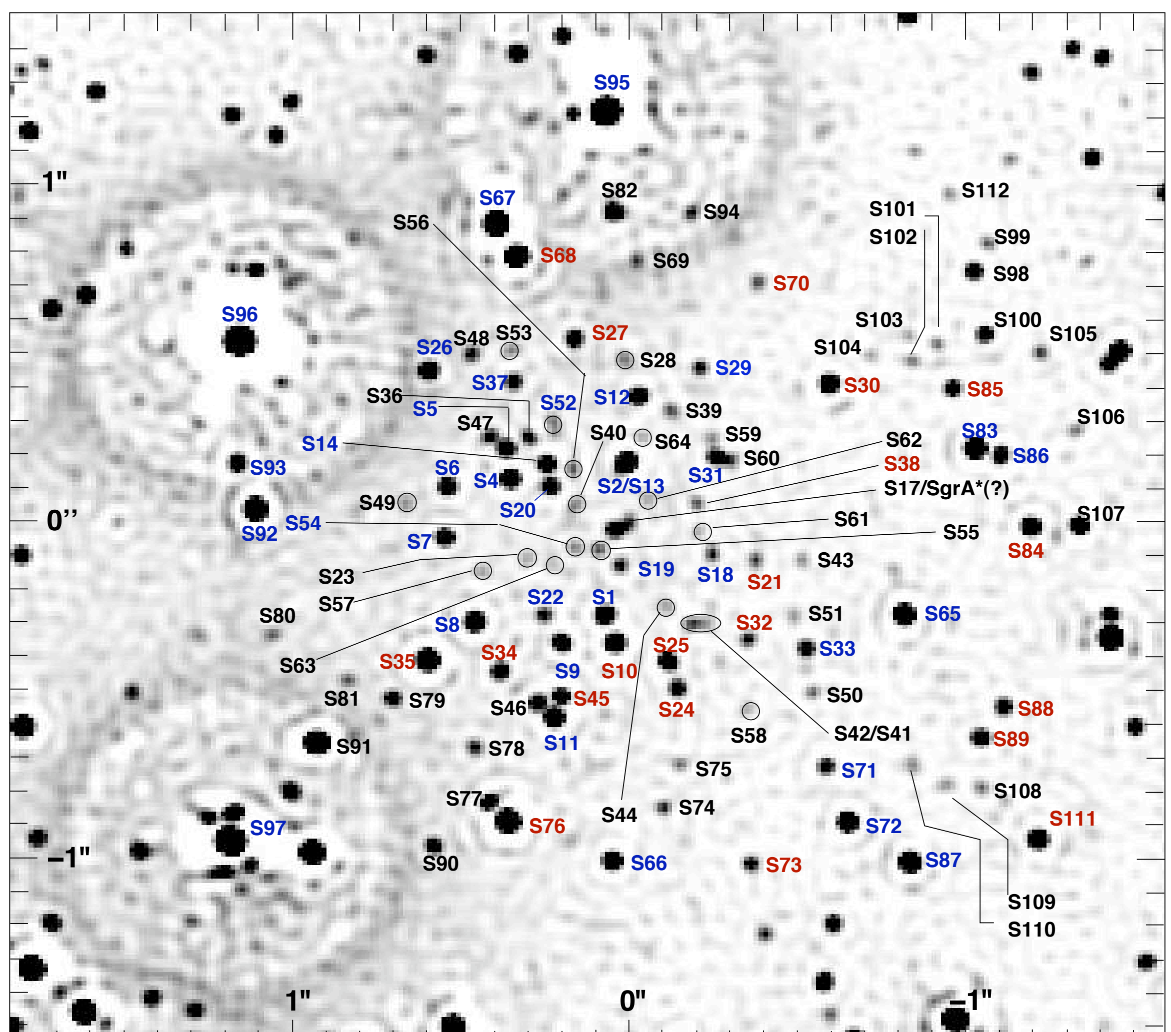
$$= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \le t \le 22.5$$

$$a(16) = 21.289 + 0.26130(16) + 0.016382(16)^2$$

$$= 29.664 m/s^2$$

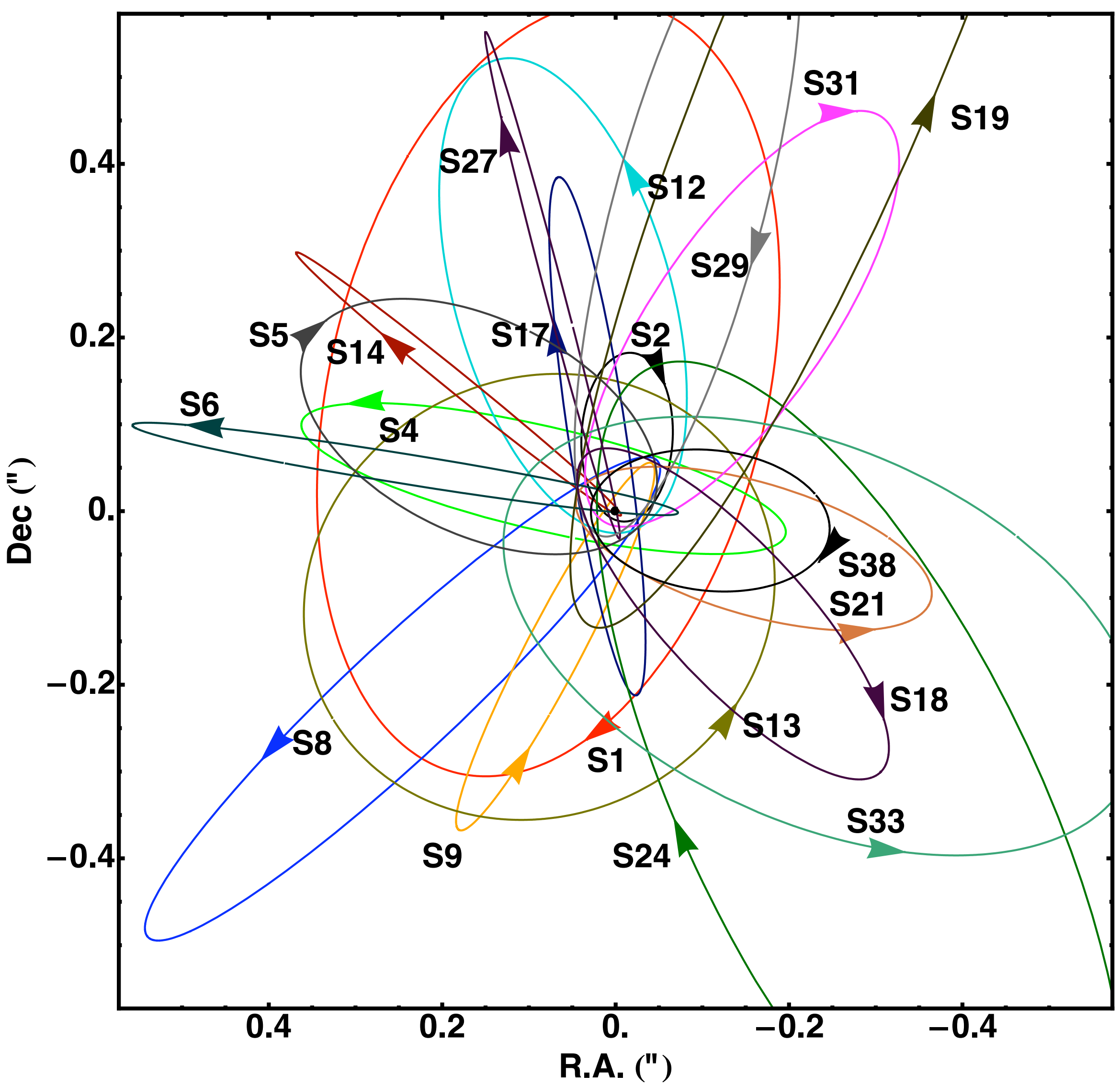
Application

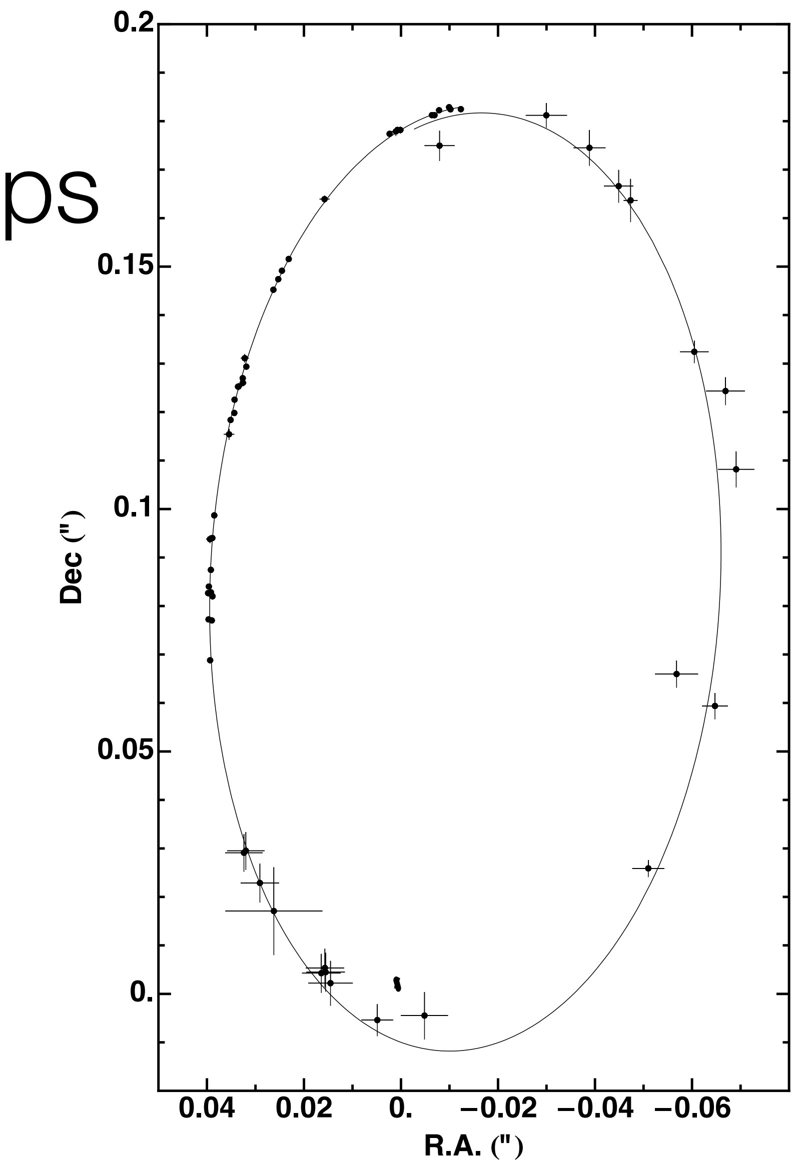
- Black hole in center of our galaxy (MBH). Basically invisible (Sgr A*).
- But stars circle MBH.

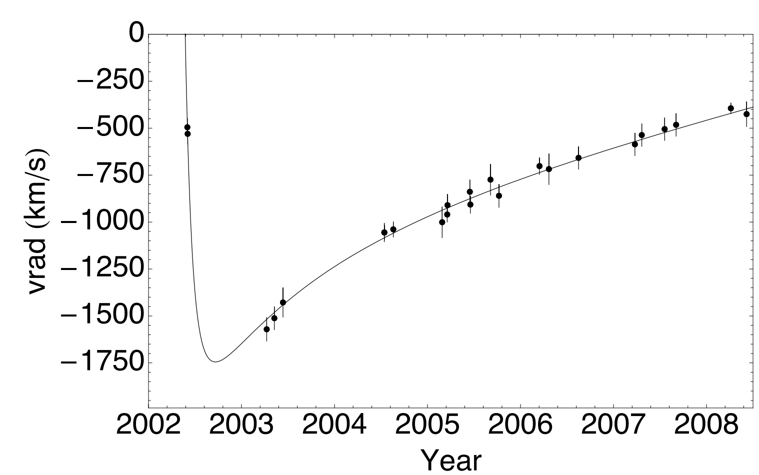


Application II

- Motion of stars (Newton, or GR) helps derive a mass for MBH.







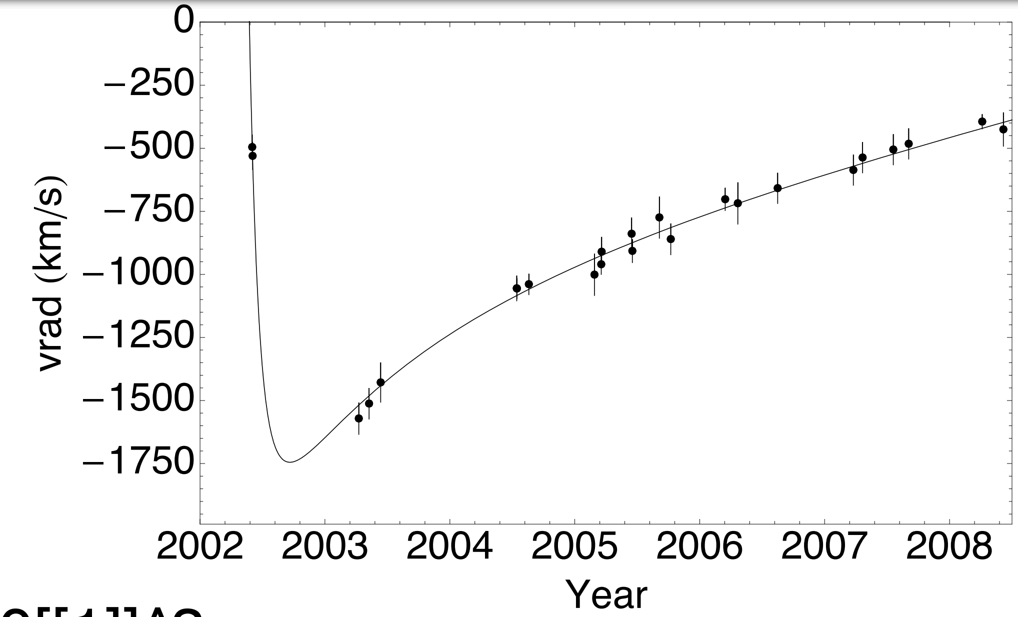
Application III

- Motion of S2

(1,1500), (2,1000), (3,800), (4,700)

x0 = {1, 2, 3, 4}

c = {1500, 1000, 800, 700}




```
Solve[{c[[1]] == a0 + a1*x0[[1]] + a2*x0[[1]]^2 + a3*x0[[1]]^3,
  c[[2]] == a0 + a1*x0[[2]] + a2*x0[[2]]^2 + a3*x0[[2]]^3,
  c[[3]] == a0 + a1*x0[[3]] + a2*x0[[3]]^2 + a3*x0[[3]]^3,
  c[[4]] == a0 + a1*x0[[4]] + a2*x0[[4]]^2 + a3*x0[[4]]^3}, {a0, a1, a2, a3}]
```

```
{{a0 -> 2500, a1 -> -(3950/3), a2 -> 350, a3 -> -(100/3)}}
```

$$
\begin{aligned}
M = & (3.95 \pm 0.06|_{\mathrm{stat}} \pm 0.18|_{\mathrm{R_0,\ stat}} \pm 0.31|_{\mathrm{R_0,\ sys}}) \\
& \times 10^6\, M_\odot \times (R_0/8\,\mathrm{kpc})^{2.19} \\
& = (4.31 \pm 0.36) \times 10^6\, M_\odot \text{ for } R_0 = 8.33\,\mathrm{kpc} \\
R_0 = & 8.33 \pm 0.17|_{\mathrm{stat}} \pm 0.31|_{\mathrm{sys}}\,\mathrm{kpc} \qquad (30)
\end{aligned}
$$

3 Divided Difference Method

Ansatz

Newton base for Polynomials:

$$N_0(x) = 1 \qquad N_i(x) = \prod_{j=0}^{i-1}(x - x_j) = (x - x_0)\cdots(x - x_{i-1}) \qquad i = 1, \ldots, n$$

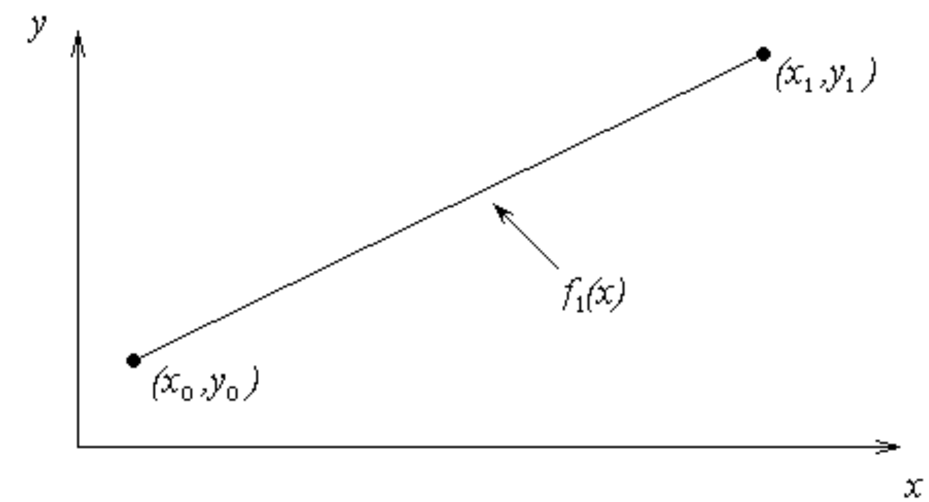
New coefficients b_i can easily determined via divided differences:

$$f_n(x) = b_0 + b_1(x - x_0) + ... + b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

Differences method, linear

- Ansatz:

$$f_1(x) = b_0 + b_1(x - x_0)$$



$$(x_0, y_0), (x_1, y_1),$$

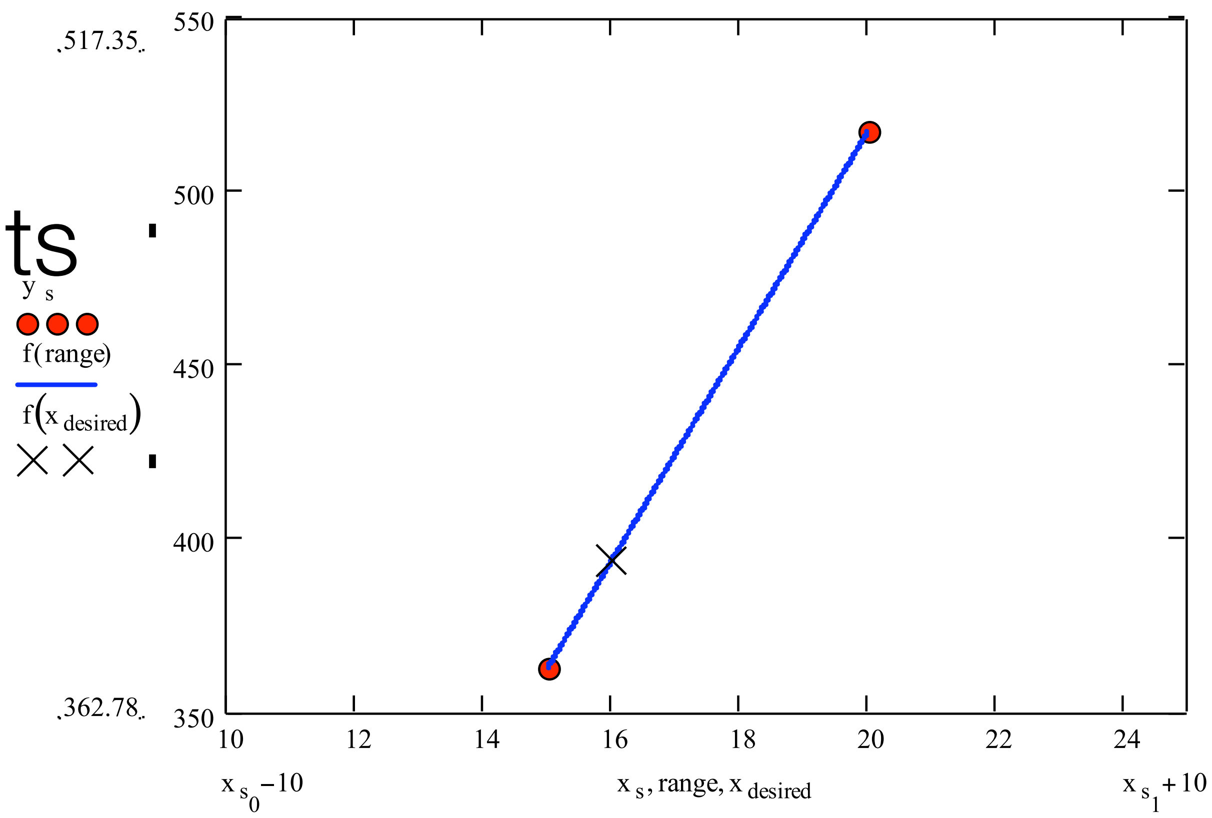
- Coefficients

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

- Ansatz: $v(t) = b_0 + b_1(t - t_0)$
- Bracketing Values and Coefficients



$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

$$b_0 = v(t_0) \quad = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} \quad = 30.914$$

- Thus

$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Difference method, quadratic

- Ansatz:

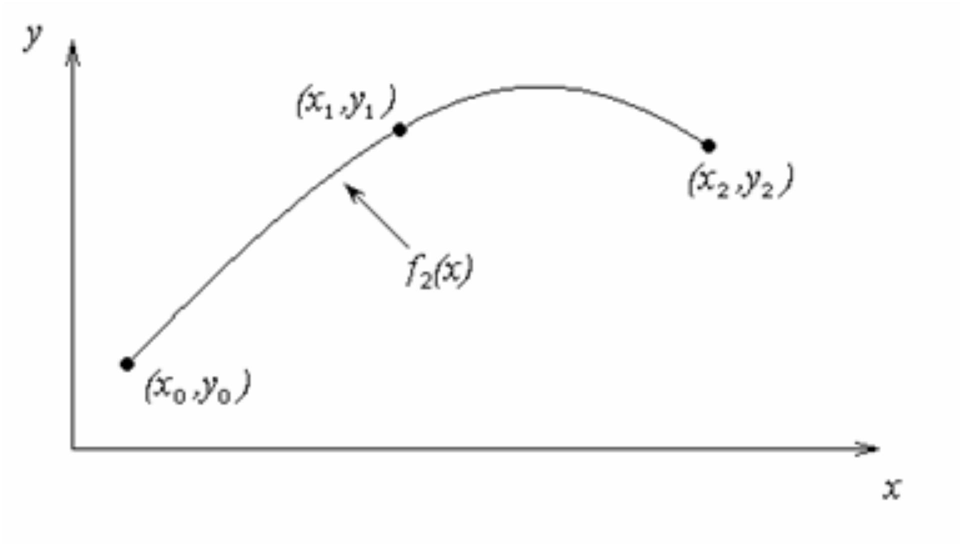
$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- Coefficients

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Evaluation of b via condition $\forall\, x_i\ f(x_i) = y_i$

Example

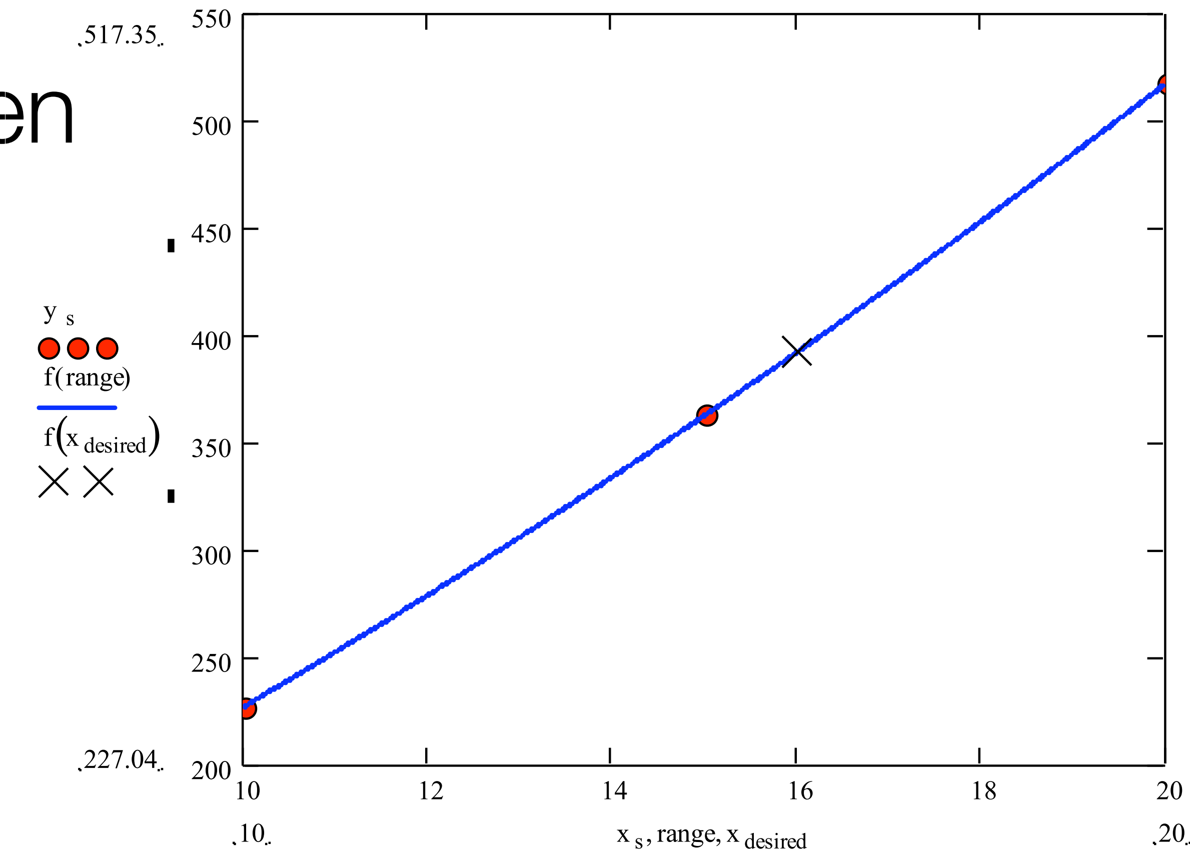
- Ansatz: $f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$

- Bracketing Values and Koeffizienten

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$



$$b_0 = v(t_0)$$
$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$
$$= 27.148$$

$$b_2 = \frac{\frac{v(t_2) - v(t_1)}{t_2 - t_1} - \frac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\frac{517.35 - 362.78}{20 - 15} - \frac{362.78 - 227.04}{15 - 10}}{20 - 10}$$
$$= \frac{30.914 - 27.148}{10}$$
$$= 0.37660$$

Example II

- Thus

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \ = 392.19 \ \text{m/s}$$

- Error estimate 2nd vs. 1st order

$$|\in_a| = \left| \frac{392.19 - 393.69}{392.19} \right| \text{x}100$$

$$= 0.38502\ \%$$

(As before with direct method!)

Difference method quadratic II

- Ansatz: $f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$

- Define new Notation:

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

- Then

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

Difference method, general

- N+1 Data points: $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$
- General Form

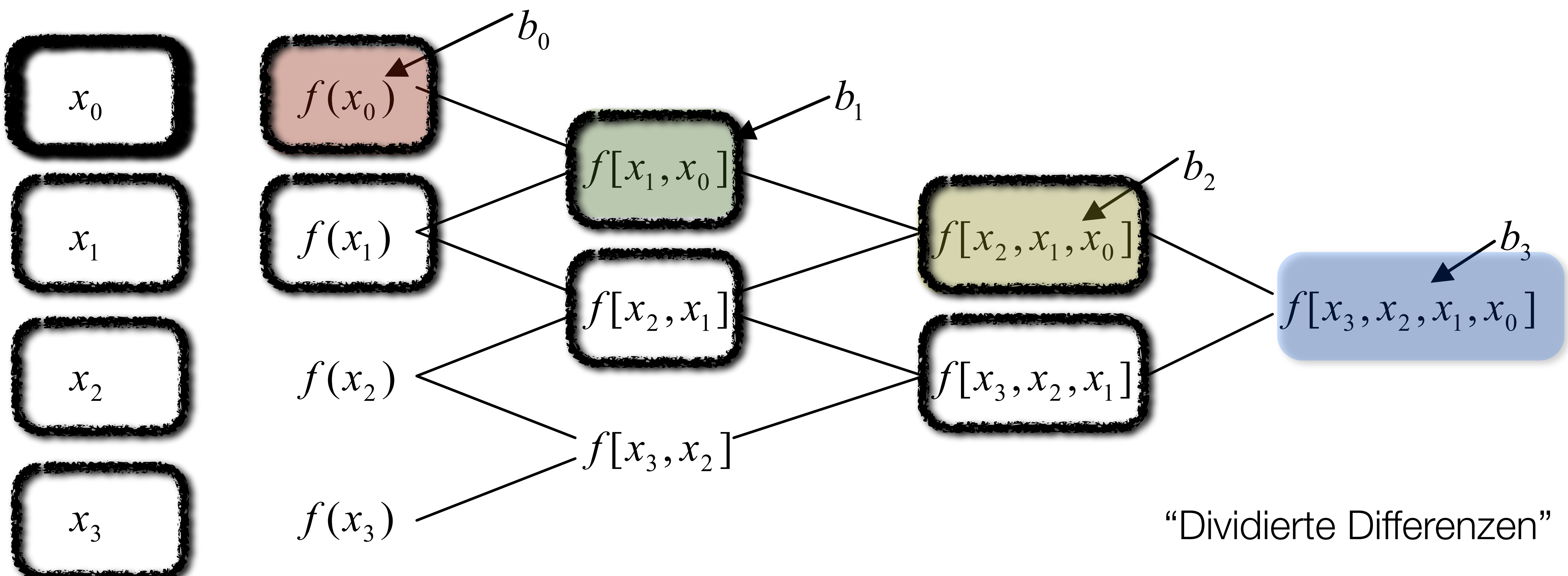
$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

- With

$$\begin{aligned}
b_0 &= f[x_0] \\
b_1 &= f[x_1, x_0] \\
b_2 &= f[x_2, x_1, x_0] \\
&\vdots \\
b_{n-1} &= f[x_{n-1}, x_{n-2}, \ldots, x_0] \\
b_n &= f[x_n, x_{n-1}, \ldots, x_0]
\end{aligned}$$

Recursion formula after Aitken

- Cubic: $$f_3(x) = \boxed{f[x_0]} + \boxed{f[x_1,x_0]}(x-x_0) + \boxed{f[x_2,x_1,x_0]}(x-x_0)(x-x_1) + \boxed{f[x_3,x_2,x_1,x_0]}(x-x_0)(x-x_1)(x-x_2)$$



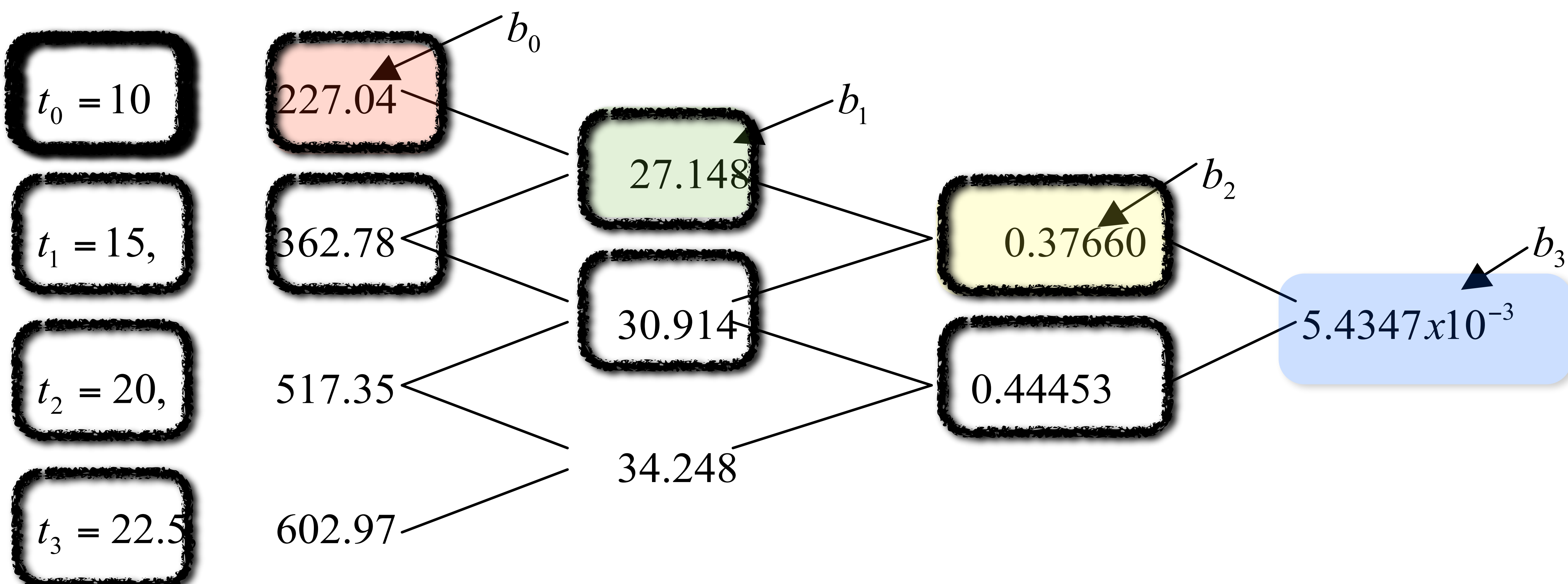
"Dividierte Differenzen"

- Upper row are the b Coefficients!
- Numerically solve one collumn after the other
- 1 more point: (higher Order), means additional row

Example

- Ansatz (cubic): $v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$
- Bracketing values

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$



$b_0 = 227.04$; $b_1 = 27.148$; $b_2 = 0.37660$; $b_3 = 5.4347*10^{-3}$

Im Beispiel II

- Thus

$$
\begin{aligned}
v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2) \\
&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\
&\quad + 5.4347 * 10^{-3}(t - 10)(t - 15)(t - 20)
\end{aligned}
$$

- At t=16s

$$
\begin{aligned}
v(16) &= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\
&\quad + 5.4347 * 10^{-3}(16 - 10)(16 - 15)(16 - 20) \\
&= 392.06 \text{ m/s}
\end{aligned}
$$

- Error estimate

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	----------	0.38502 %	0.033427 %

As before…

3 Splines

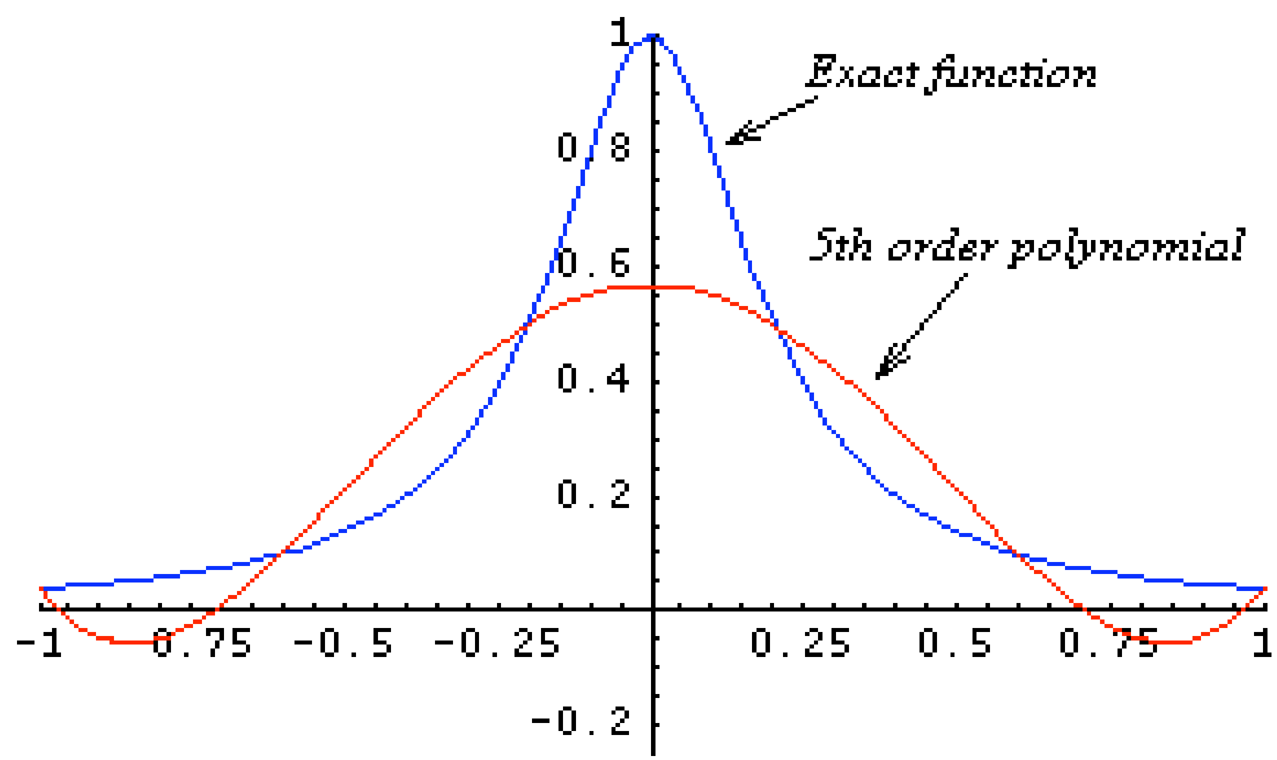
Why Splines?

- Assume (simple) Funktion

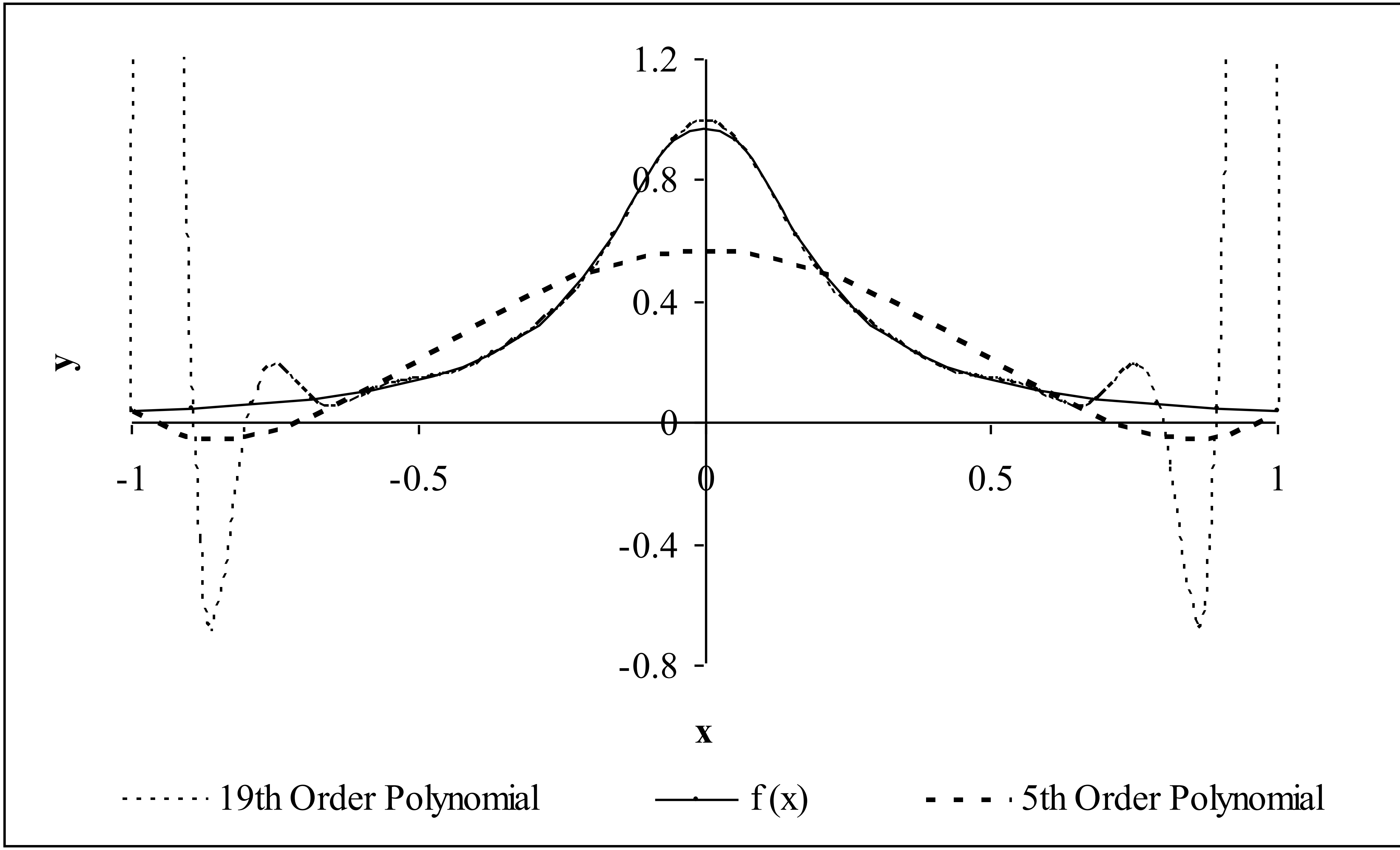
$$f(x) = \frac{1}{1+25x^2} \qquad \text{(Runges Function)}$$

- Six equidistant points im Interval [-1,1]
- Interpolation With Polynomial 5th Order.

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461



Higher order does not help!



Idea Splines

- Only use a few points x_i around x
  - No Oscillations

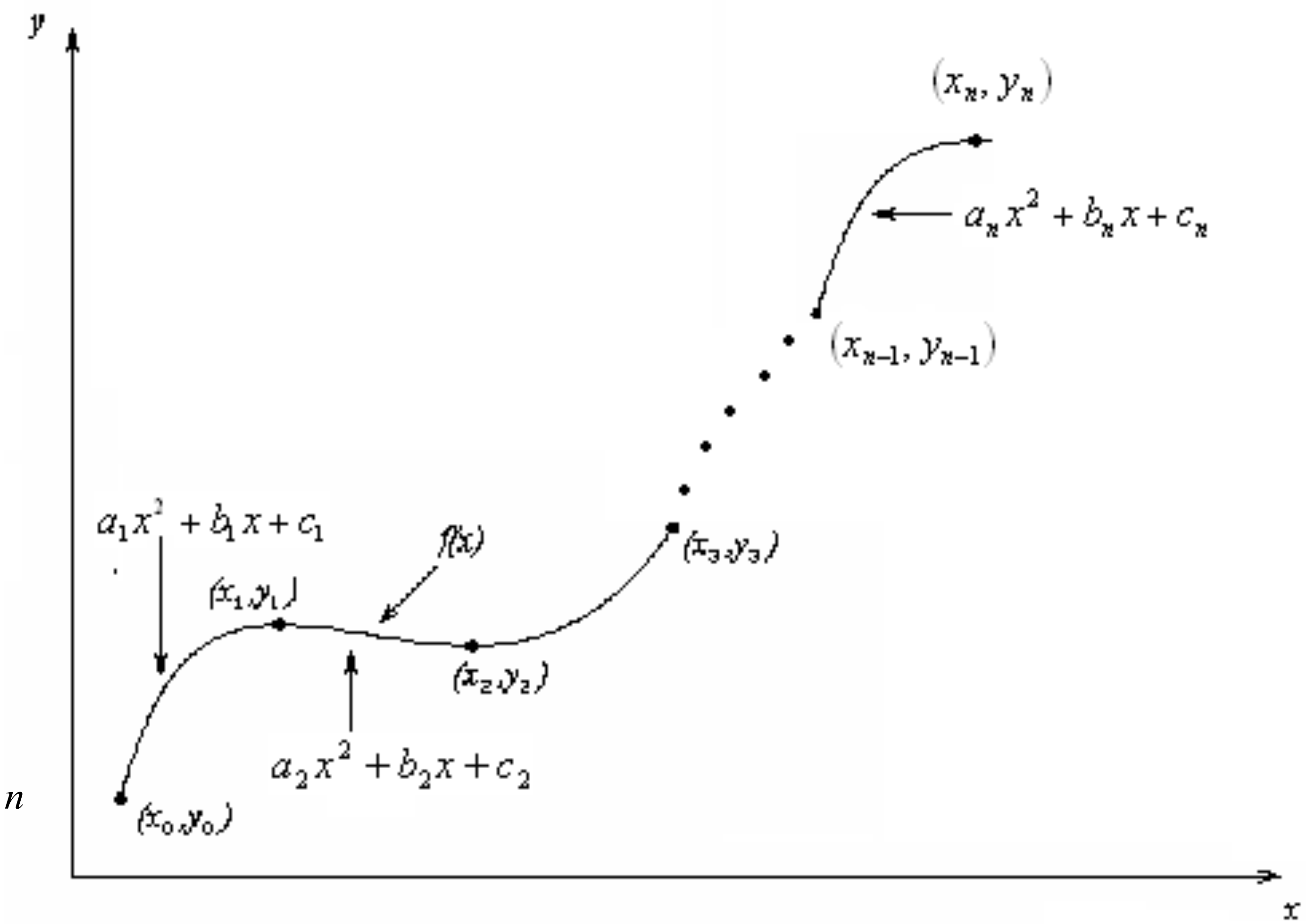
- But use values outside interval
  - Ask for continuous derivatives!

- Trivial Case: Sequential of linear Interpolations (Polygon)

Quadratic Splines

- Quadratic interpolation between neighbours plus additional condition. $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, && x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, && x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



- Needed: $a_i, b_i, c_i, i = 1, 2, \ldots, n$

Bedingung 1

- Interpolant is to be continuous:
  - Each spline connects two data points

$$a_1 x_0^{\,2} + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^{\,2} + b_1 x_1 + c_1 = f(x_1) \quad .$$

$$.$$

$$.$$

$$a_i x_{i-1}^{\,2} + b_i x_{i-1} + c_i = f(x_{i-1})$$

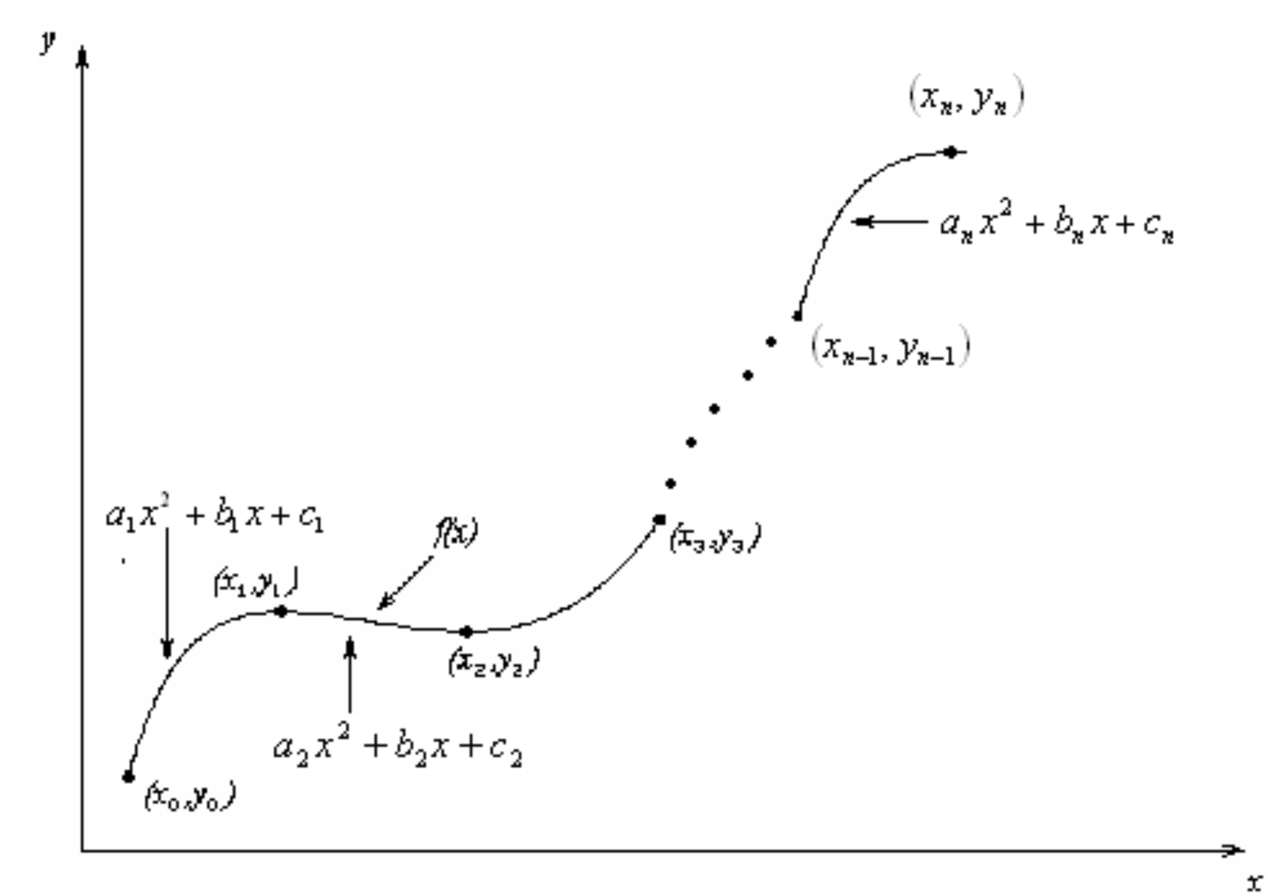
$$a_i x_i^{\,2} + b_i x_i + c_i = f(x_i) \quad .$$

$$.$$

$$.$$

$$a_n x_{n-1}^{\,2} + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^{\,2} + b_n x_n + c_n = f(x_n)$$



- Lead to 2N Equations

Condition 2

- First derivative shall be continuous from one segment to the next:

For example, the derivative of the first spline

$$a_1 x^2 + b_1 x + c_1 \quad \text{is} \qquad 2a_1 x + b_1$$

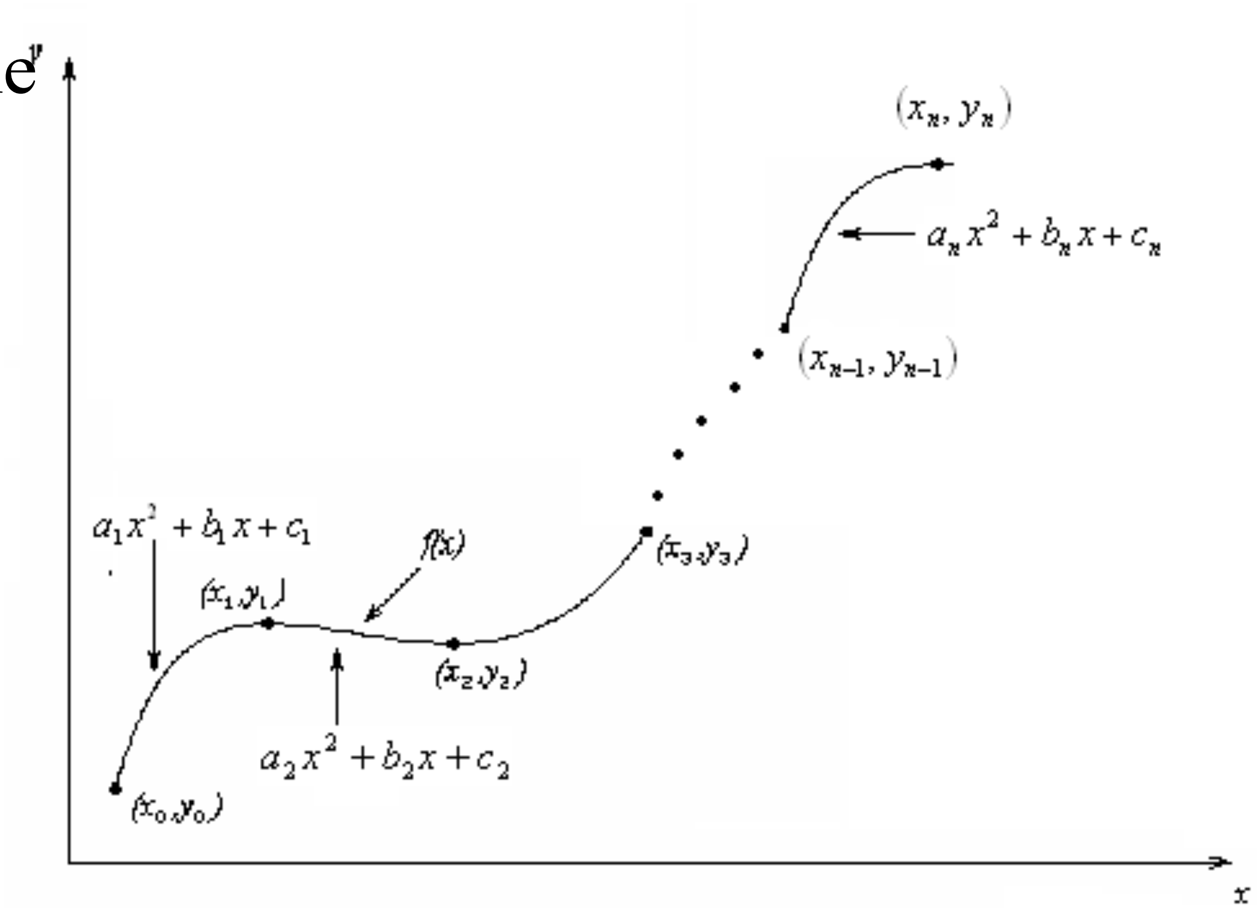
The derivative of the second spline

$$a_2 x^2 + b_2 x + c_2 \quad \text{is} \qquad 2a_2 x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$$

$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$



Conditon 2 II

- Same condition for all splines:

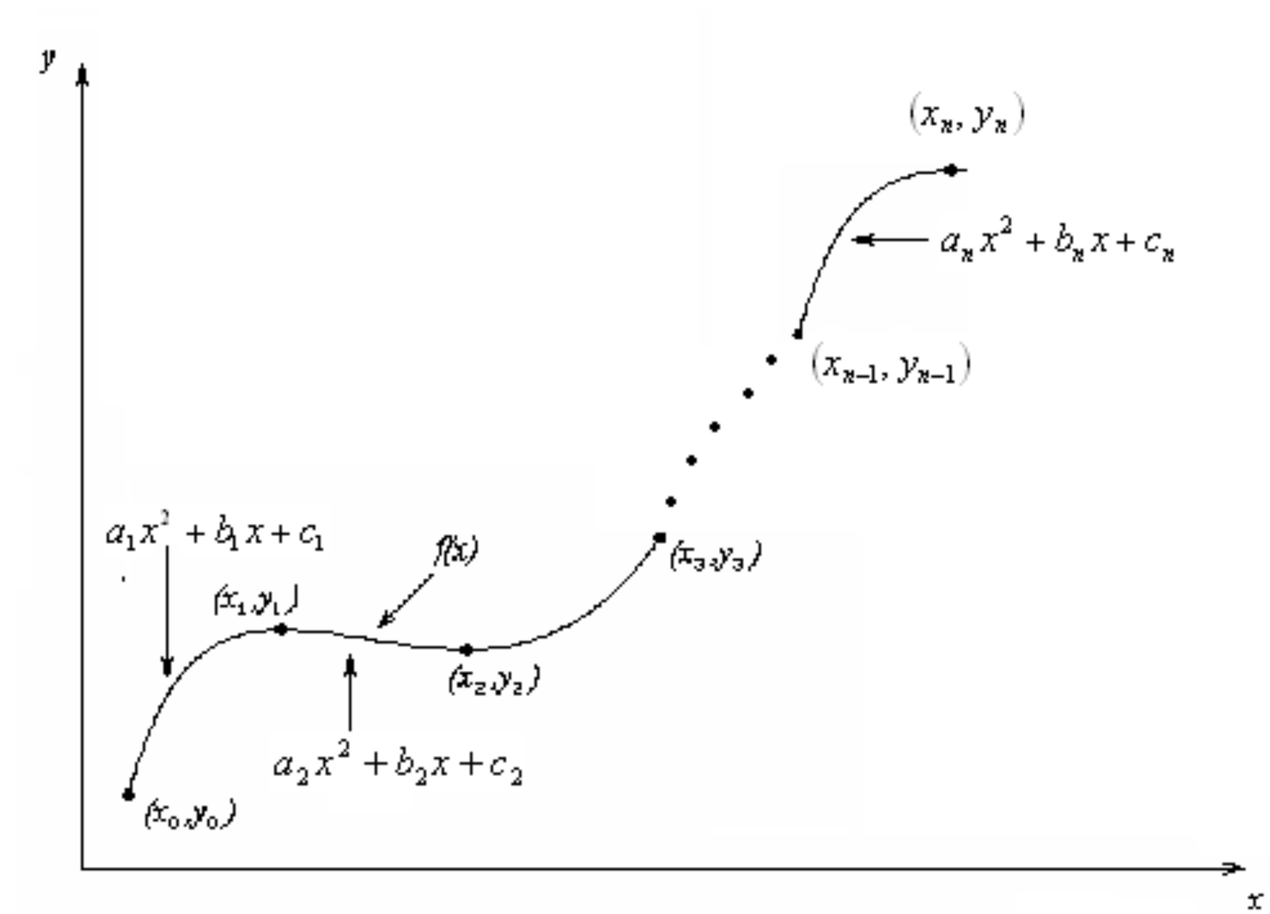
$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

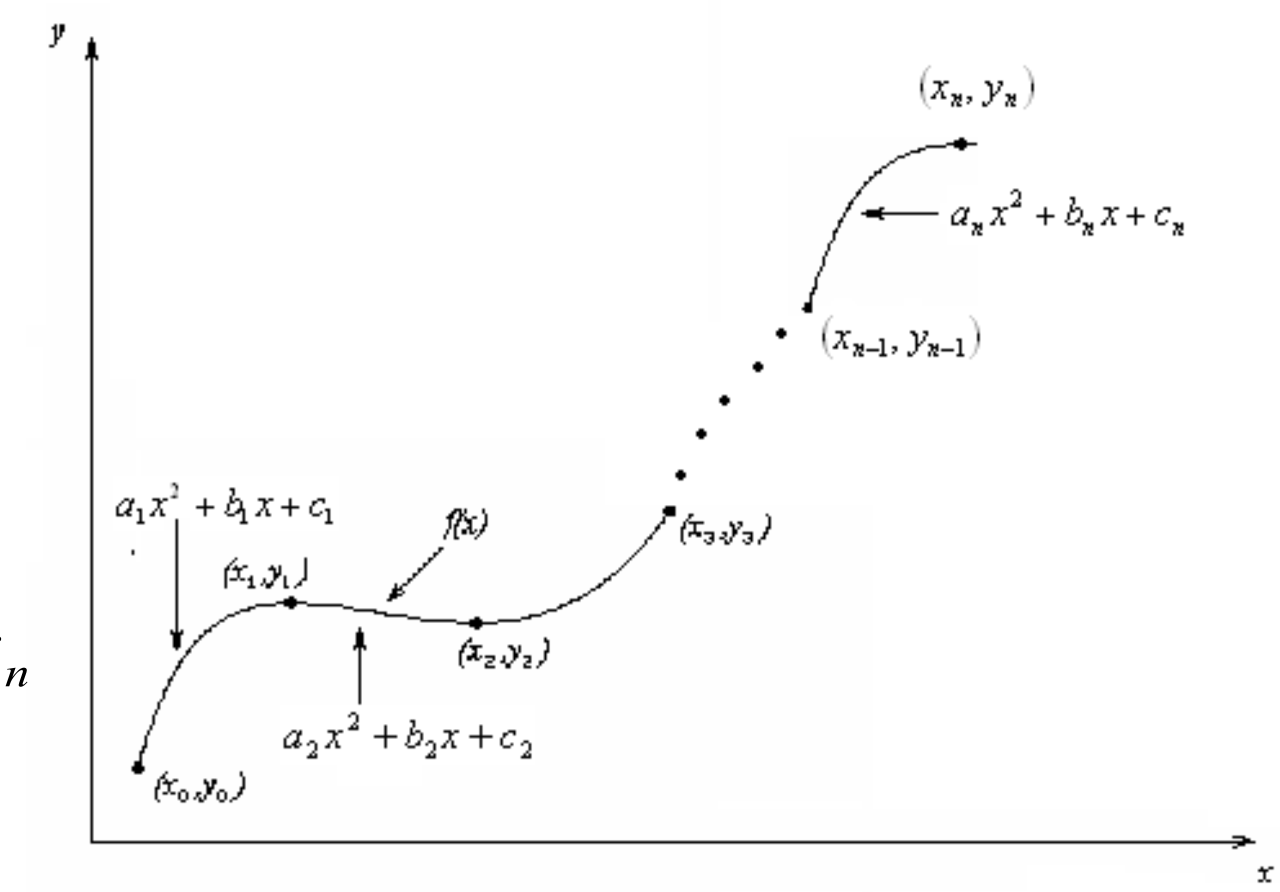


- Thus additional N-1 Equations
- and in total 2 N + N -1 = 3 N -1Equations. (1 missing?)
- Final assumption: first spline is linear: a_1=0

Derivation

- Finally we have 3 N equations for 3 N unknowns.

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&. \\
&. \\
&. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



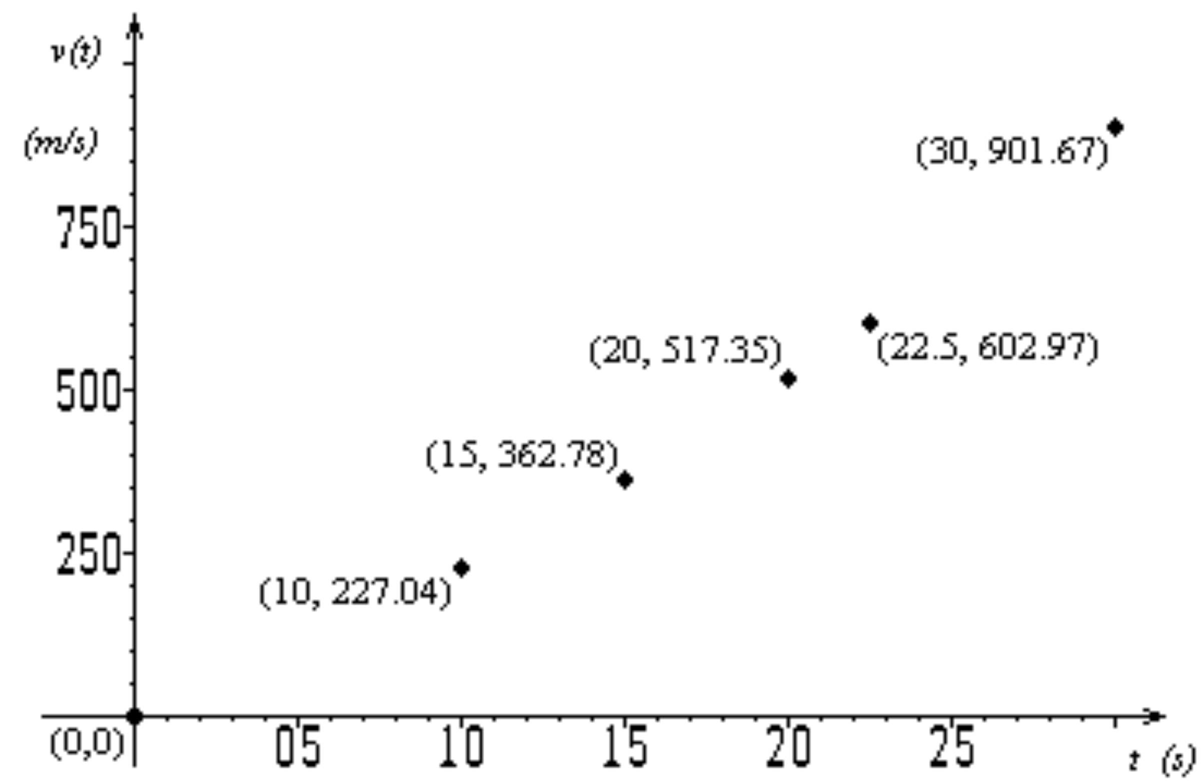
- Determine: a_i, b_i and c_i
- Spline Interpolation for full interval defined
- Sequential quadratic interpolations fulfilling additional conditions, continuous in derivatives

Example

- With 6 data points

We need 5 Splines

$$
\begin{aligned}
v(t) &= a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10 \\
&= a_2 t^2 + b_2 t + c_2, \quad 10 \le t \le 15 \\
&= a_3 t^2 + b_3 t + c_3, \quad 15 \le t \le 20 \\
&= a_4 t^2 + b_4 t + c_4, \quad 20 \le t \le 22.5 \\
&= a_5 t^2 + b_5 t + c_5, \quad 22.5 \le t \le 30
\end{aligned}
$$

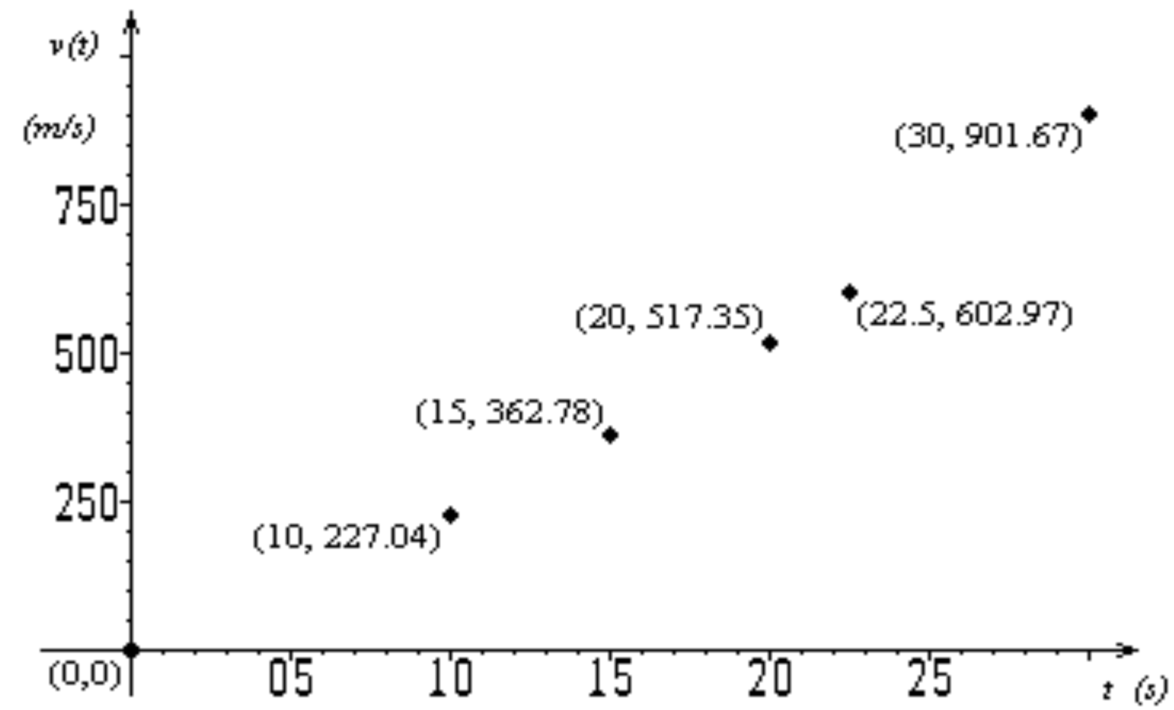


Example II

- Generate system of equations: condition 1:
- Spline 1 connects x_0 and x_1

$$a_1(0)^2 + b_1(0) + c_1 = 0 \qquad (1)$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04 \qquad (2)$$



- Analogue:

$$a_2(10)^2 + b_2(10) + c_2 = 227.04 \qquad (3)$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78 \qquad (4)$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78 \qquad (5)$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35 \qquad (6)$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35 \qquad (7)$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97 \qquad (8)$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97 \qquad (9)$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67 \qquad (10)$$

Example III

- Generate system of equations: condition 2 (Derivatives)

At $t = 10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0 \qquad (11)$$

At $t = 15$

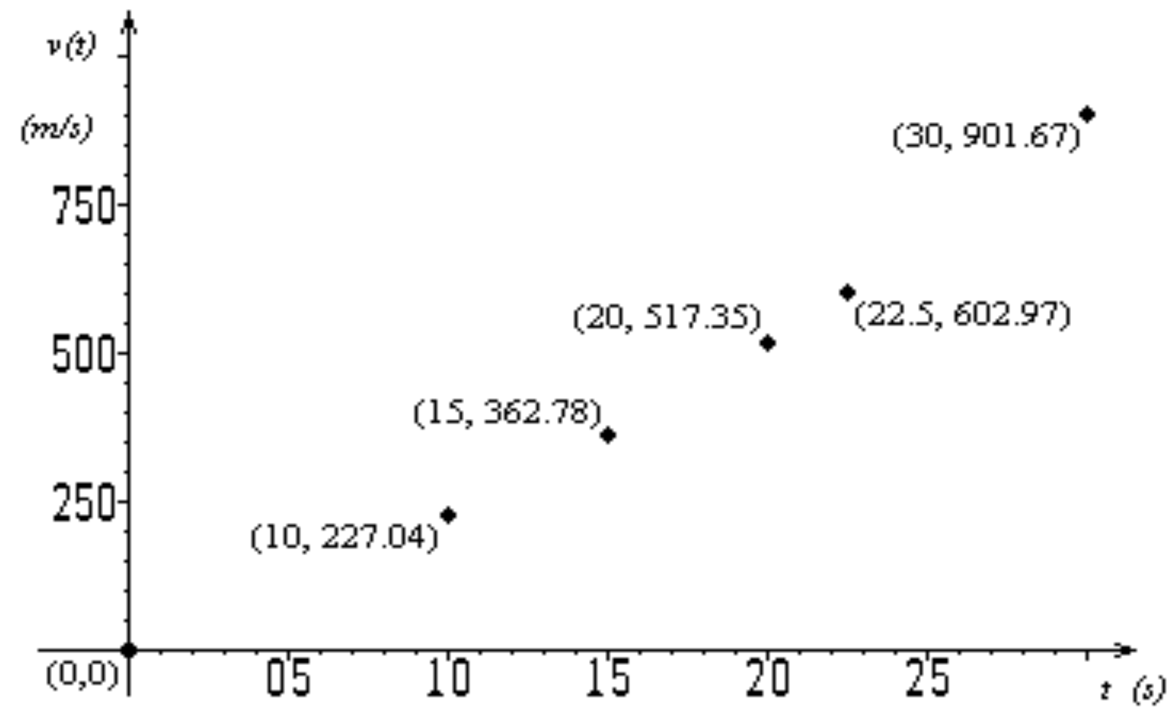
$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0 \qquad (12)$$

At $t = 20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0 \qquad (13)$$

At $t = 22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0 \qquad (14)$$



- Additional assumption : $a_1 t^2 + b_1 t + c_1$ is linear,

$$a_1 = 0 \qquad (15)$$

Im Beispiel IV

- *Complete System* (as Matrix)

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

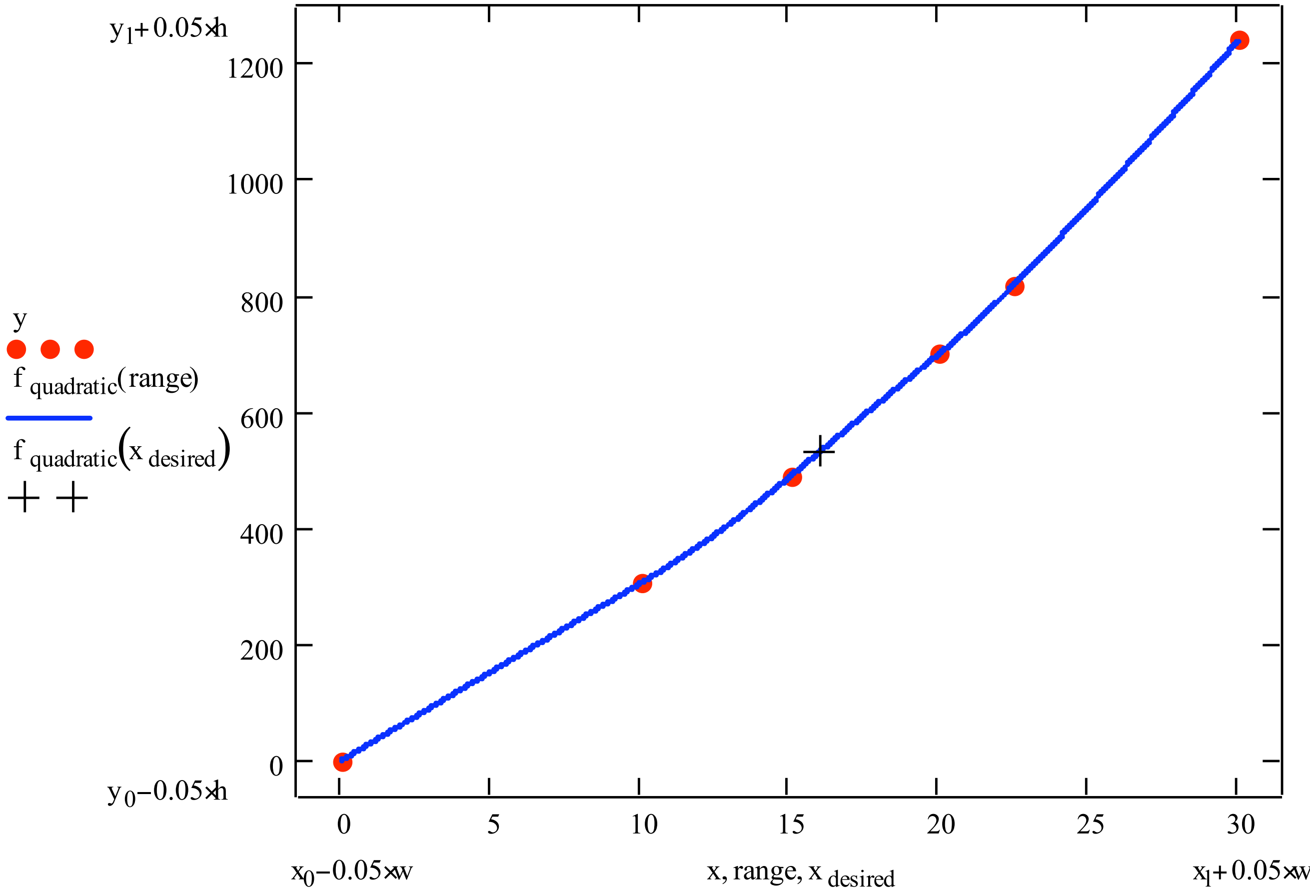
- Results in:

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	-0.1356	35.66	-141.61
4	1.6048	-33.956	554.55
5	0.20889	28.86	-152.13

Example V

- Final result:

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$



- For t=16 s you get

$$v(16) = -0.1356(16)^2 + 35.66(16) - 141.61 = 394.24 \text{ m/s}$$

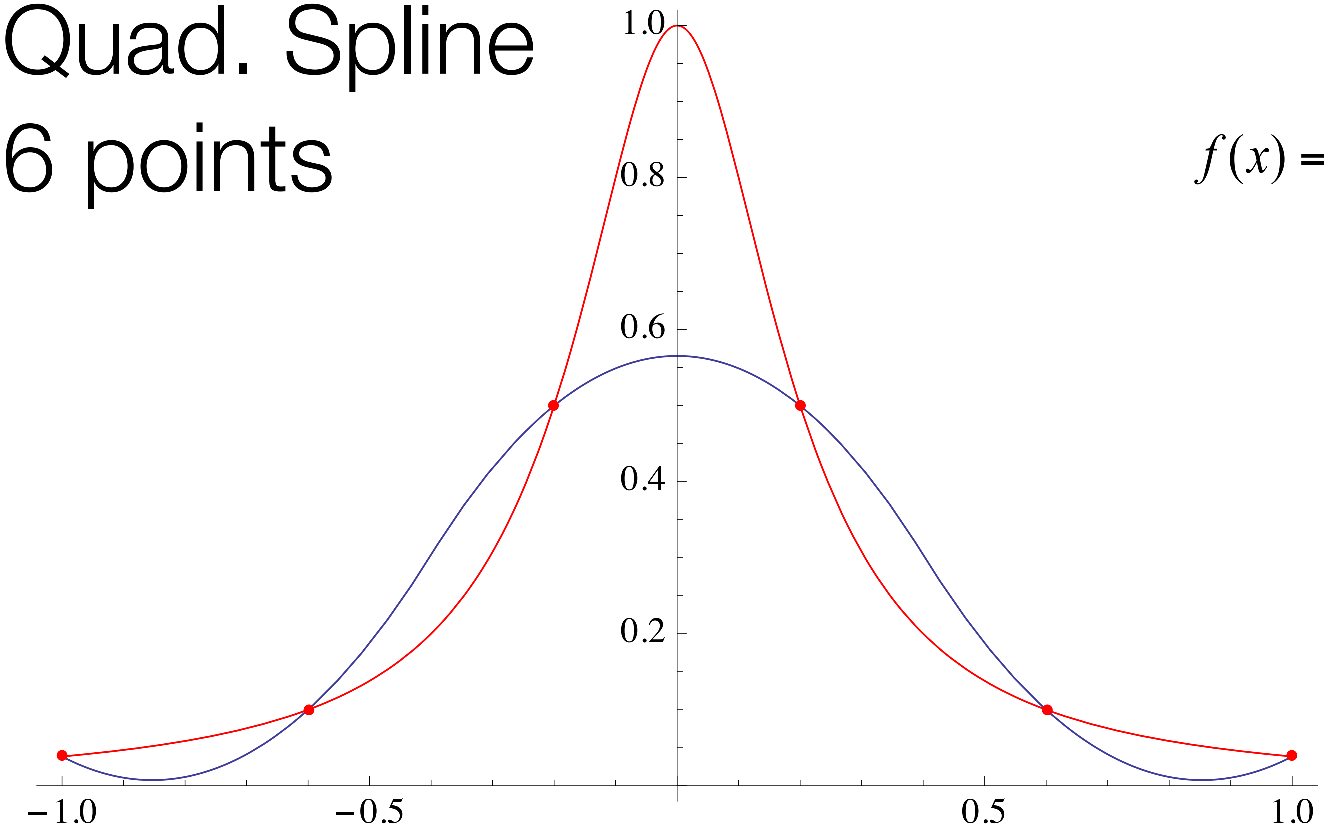
- *Error Estimate quad spline* vs linear:
- better than 3rd vs 2nd order polynomial

$$|\in_a| = \left|\frac{394.24 - 393.7}{394.24}\right| \times 100$$

$$= 0.1369\%$$

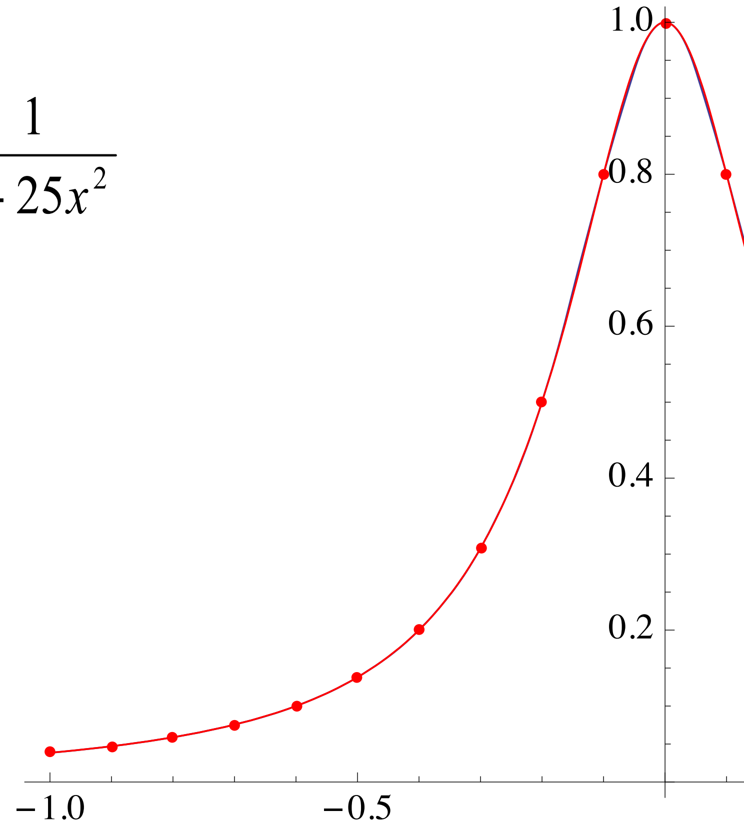
Comparison Splines vs. Polynomials

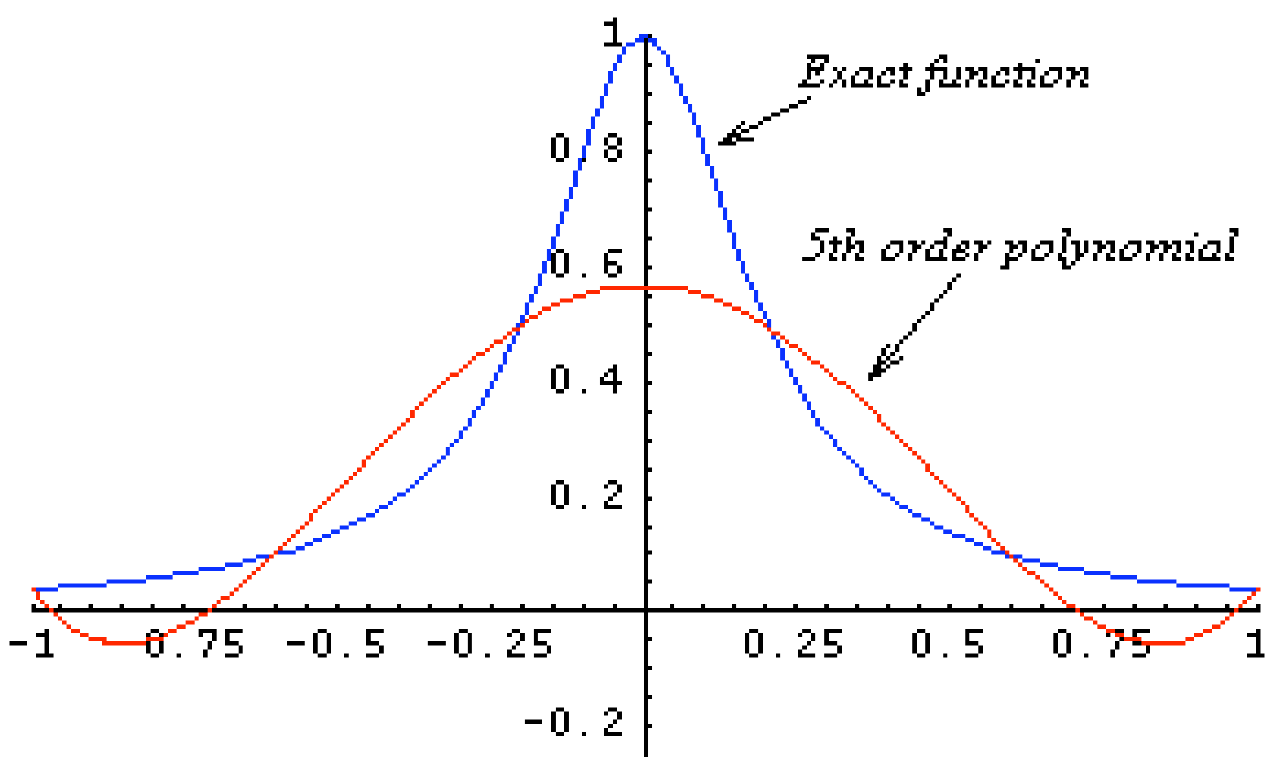
Quad. Spline
6 points

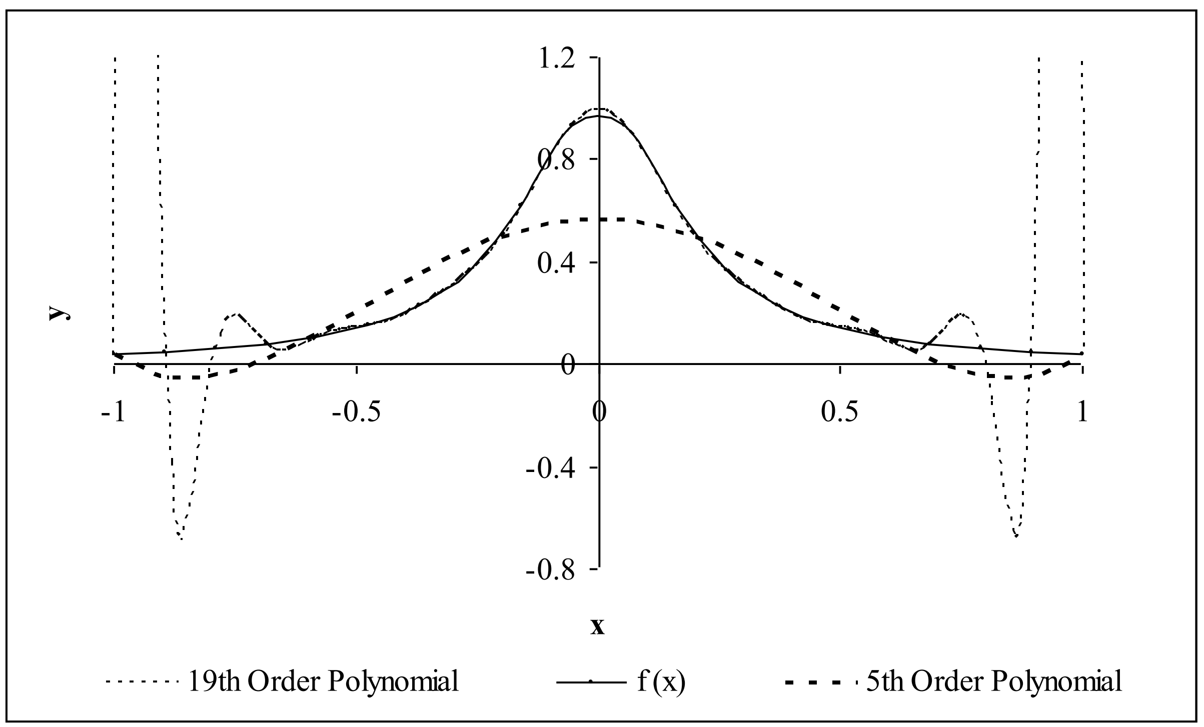


$$f(x) = \frac{1}{1+25x^2}$$

Quad. Spline
21 points







Ressources

- Based on: http://numericalmethods.eng.usf.edu
By Autar Kaw, Jai Paul

- Strongly suggested reading material for reference:
Numerical Recipes (2nd/3rd Edition). Press et al., Cambridge University Press
http://www.nr.com/oldverswitcher.html

- **Exercise 1, 10 points**: Newton's Divided Differences for known polynomial

  Write a program that computes the Newtonian divided differences for five points (i.e. up to the fourth divided difference should be used). To test your program, use the polynomial $y = f(x) = 0.1x^4 - x^2$ to compute 5 pairs of points (x, y), $(x = 0, 0.3, 13.0, -4.8, -9.0)$ and compare the original polynomial with the interpolation polynomial by plotting their values (and check the difference).

- **Exercise 2, 10 points**:
  Use your program up to the third divided difference to obtain a fitting polynomial for the four points given (motion of the S2 star in the galactic centre). Compare its results with the direct interpolation polynomial derived in the lecture.

  The data points to be used are (1,1500), (2,1000), (3,800), (4,700), in units of years and km/s. They are approximately taken from the following plot of measurements obtained by Gillessen et al. (2008), with the year 1 defined as the mid-point between 2003 and 2004.