Holistic Numerical Methods Institute

Integration

Topic: Trapezoidal Rule

Major: General Engineering

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What is Integration

Integration:

The process of measuring the area under a function plotted on a graph.

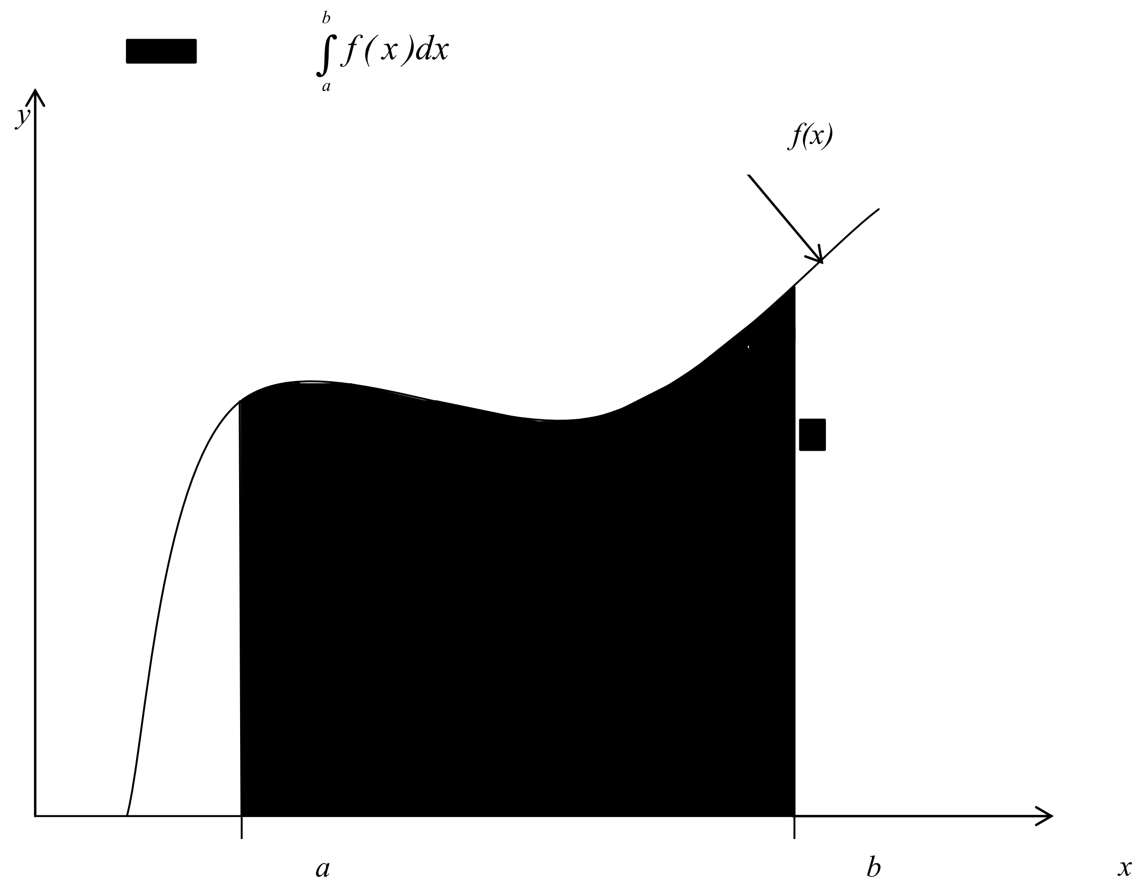
$$I = \int_a^b f(x)\,dx$$

Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration



Basis of Trapezoidal Rule

Trapezoidal Rule is based on the Newton-Cotes Formula that states if one can approximate the integrand as an n^{th} order polynomial…

$$I = \int_a^b f(x)dx$$ where $$f(x) \approx f_n(x)$$

and $$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$$

Basis of Trapezoidal Rule

Then the integral of that function is approximated by the integral of that n^{th} order polynomial.

$$\int_a^b f(x) \approx \int_a^b f_n(x)$$

Trapezoidal Rule assumes n=1, that is, the area under the linear polynomial,

$$\int_a^b f(x)dx \quad = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Derivation of the Trapezoidal Rule

Method Derived From Geometry

The area under the curve is a trapezoid. The integral

$$\int_a^b f(x)dx \approx Area\ of\ trapezoid$$

$$= \frac{1}{2}(Sum\ of\ parallel\ sides)(height)$$

$$= \frac{1}{2}\left(f(b)+f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

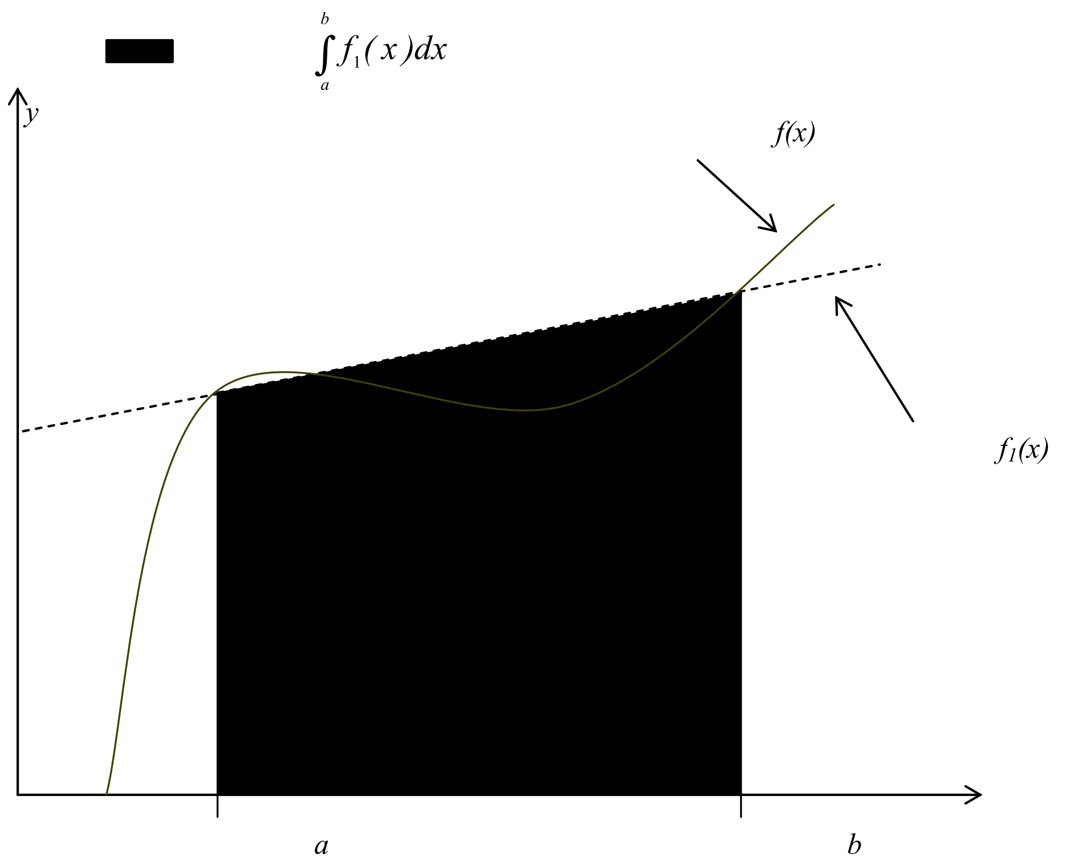


Figure 2: Geometric Representation

Example 1

The vertical distance covered by a rocket from t=8 to t=30 seconds is given by:

$$x = \int_{8}^{30} \left(2000\, ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use single segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) $$I \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$a = 8 \qquad\qquad b = 30$$

$$f(t) = 2000\,ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000\,ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) \qquad = 177.27\ m/s$$

$$f(30) = 2000\,ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) \quad = 901.67\ m/s$$

Solution (cont)

a) $$I = (30-8)\left[\frac{177.27+901.67}{2}\right]$$

$$= 11868\ m$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt \quad = 11061\ m$$

Solution (cont)

b)

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11868$$

$$= -807\ m$$

c) The absolute relative true error, $|\in_t|$, would be

$$|\in_t| = \left|\frac{11061 - 11868}{11061}\right| \times 100 = 7.2959\%$$

Multiple Segment Trapezoidal Rule

In Example 1, the true error using single segment trapezoidal rule was large. We can divide the interval [8,30] into [8,19] and [19,30] intervals and apply Trapezoidal rule over each segment.

$$f(t) = 2000\ln\left(\frac{140000}{140000-2100t}\right) - 9.8t$$

$$\int_{8}^{30} f(t)dt = \int_{8}^{19} f(t)dt + \int_{19}^{30} f(t)dt$$

$$= (19-8)\left[\frac{f(8)+f(19)}{2}\right] + (30-19)\left[\frac{f(19)+f(30)}{2}\right]$$

Multiple Segment Trapezoidal Rule

With

$$f(8) = 177.27\ m/s$$

$$f(30) = 901.67\ m/s$$

$$f(19) = 484.75\ m/s$$

Hence:

$$\int_8^{30} f(t)dt = (19-8)\left[\frac{177.27+484.75}{2}\right] + (30-19)\left[\frac{484.75+901.67}{2}\right]$$

$$= 11266\ m$$

Multiple Segment Trapezoidal Rule

The true error is:

$$E_t = 11061 - 11266$$

$$= -205\ m$$

The true error now is reduced from -807 m to -205 m.

Extending this procedure to divide the interval into equal segments to apply the Trapezoidal rule; the sum of the results obtained for each segment is the approximate value of the integral.

Multiple Segment Trapezoidal Rule

Divide into equal segments as shown in Figure 4. Then the width of each segment is:

$$h = \frac{b-a}{n}$$

The integral I is:

$$I = \int_a^b f(x)dx$$

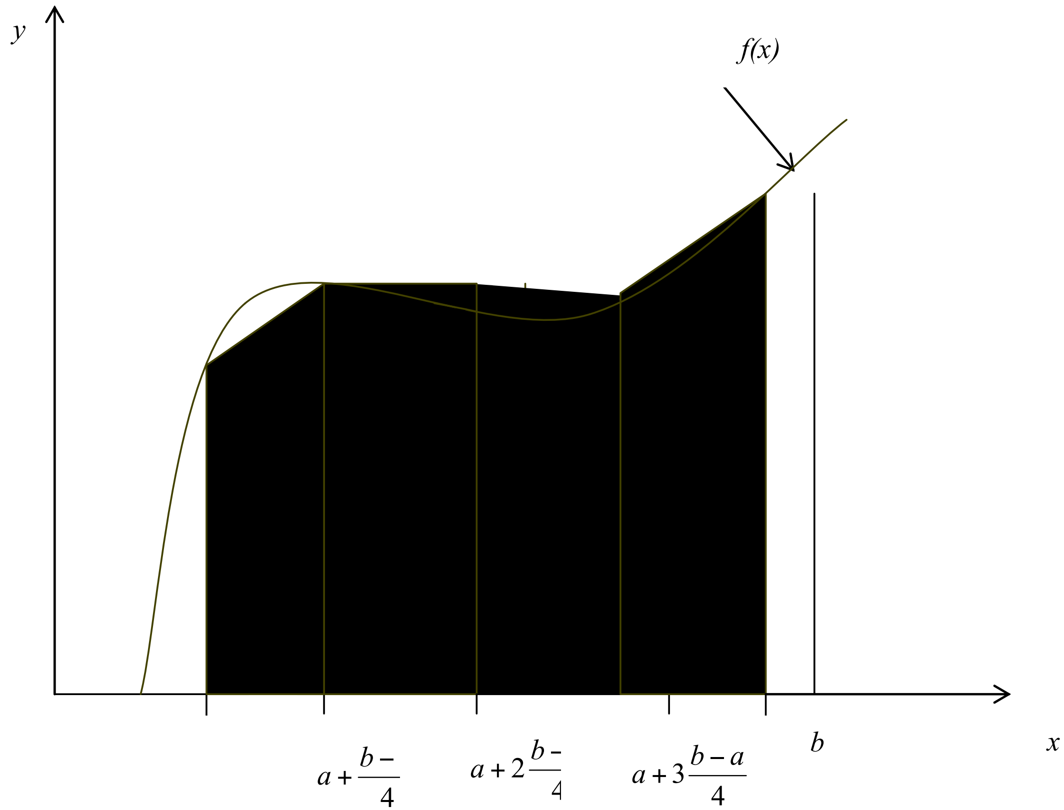


Figure 4: Multiple (n=4) Segment Trapezoidal Rule

Multiple Segment Trapezoidal Rule

The integral I can be broken into h integrals as:

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

Applying Trapezoidal rule on each segment gives:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

Example 2

The vertical distance covered by a rocket from to seconds is given by:

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use two-segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $\left| \varepsilon_a \right|$ for part (a).

Solution

a) The solution using 2-segment Trapezoidal rule is

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$n = 2 \qquad a = 8 \qquad b = 30$$

$$h = \frac{b-a}{n} = \frac{30-8}{2} = 11$$

Solution (cont)

Then:

$$I = \frac{30-8}{2(2)}\left[f(8) + 2\left\{\sum_{i=1}^{2-1} f(a+ih)\right\} + f(30)\right]$$

$$= \frac{22}{4}\left[f(8) + 2f(19) + f(30)\right]$$

$$= \frac{22}{4}\left[177.27 + 2(484.75) + 901.67\right]$$

$$= 11266\ m$$

Solution (cont)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \quad = 11061\ m$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061 - 11266$$

Solution (cont)

c) The absolute relative true error, $|\in_t|$, would be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{11061 - 11266}{11061}\right| \times 100$$

$$= 1.8534\%$$

Solution (cont)

Table 1 gives the values obtained using multiple segment Trapezoidal rule for:

$$x = \int_8^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Table 1: Multiple Segment Trapezoidal Rule Values

Example 3

Use Multiple Segment Trapezoidal Rule to find the area under the curve

$$f(x) = \frac{300x}{1+e^{x}} \quad \text{from} \quad x = 0 \quad \text{to} \quad x = 10$$

Using two segments, we get $h = \dfrac{10-0}{2} = 5$ and

$$f(0) = \frac{300(0)}{1+e^{0}} = 0 \qquad f(5) = \frac{300(5)}{1+e^{5}} = 10.039 \qquad f(10) = \frac{300(10)}{1+e^{10}} = 0.136$$

Solution

Then:

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$= \frac{10-0}{2(2)}\left[f(0) + 2\left\{\sum_{i=1}^{2-1} f(0+5)\right\} + f(10)\right]$$

$$= \frac{10}{4}\left[f(0) + 2f(5) + f(10)\right] = \frac{10}{4}\left[0 + 2(10.039) + 0.136\right]$$

$$= 50.535$$

Solution (cont)

So what is the true value of this integral?

$$\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59$$

Making the absolute relative true error:

$$|\in_t| = \left|\frac{246.59 - 50.535}{246.59}\right| \times 100\%$$

$$= 79.506\%$$

Solution (cont)

Table 2: Values obtained using Multiple Segment Trapezoidal Rule for:

$$\int_0^{10} \frac{300x}{1+e^x}\,dx$$

n	Approximate Value	E_t	$\lvert \in_t \rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Error in Multiple Segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by:

$$E_t = \frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$ where ζ is some point in $[a,b]$

What is the error, then in the multiple segment Trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment Trapezoidal rule.

The error in each segment is

$$E_1 = \frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= \frac{h^3}{12} f''(\zeta_1)$$

Error in Multiple Segment Trapezoidal Rule

Similarly:

$$E_i = \frac{\left[(a+ih)-(a+(i-1)h)\right]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= \frac{h^3}{12} f''(\zeta_i)$$

It then follows that:

$$E_n = \frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= \frac{h^3}{12} f''(\zeta_n)$$

Error in Multiple Segment Trapezoidal Rule

Hence the total error in multiple segment Trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i \quad = \frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i) \quad = \frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

.

The term $\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the $f''(x),\ a < x < b$

Hence:

$$E_t = \frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

Error in Multiple Segment Trapezoidal Rule

Below is the table for the integral $$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

n	**Value**	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.854	5.343
4	11113	-51.5	0.4655	0.3594
8	11074	-12.9	0.1165	0.03560
16	11065	-3.22	0.02913	0.00401

Integration

Topic: Simpson's 1/3rd Rule

Major: General Engineering

Basis of Simpson's 1/3rd Rule

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

Where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Basis of Simpson's 1/3rd Rule

Choose

$$(a, f(a)),\ \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \quad \text{and} \quad (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Basis of Simpson's 1/3rd Rule

Solving the previous equations for a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Basis of Simpson's 1/3rd Rule

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Basis of Simpson's 1/3rd Rule

Substituting values of a_0, a_1, a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Basis of Simpson's 1/3rd Rule

Hence

$$\int_a^b f_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

Example 1

The distance covered by a rocket from t=8 to t=30 is given by

$$x = \int_{8}^{30} \left(2000\,ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x

b) Find the true error, E_t

c) Find the absolute relative true error, $|\in_t|$

Solution

a)

$$x = \int_0^{10} f(t)dt$$

$$x = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30)\right]$$

$$= \left(\frac{22}{6}\right)\left[177.2667 + 4(484.7455) + 901.6740\right]$$

$$= 11065.72\ m$$

Solution (cont)

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

$$= 11061.34 \ m$$

True Error

$$E_t = 11061.34 - 11065.72$$

$$= -4.38 \ m$$

Solution (cont)

c) Absolute relative true error,

$$|\in_t| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\%$$

$$= 0.0396\%$$

Multiple Segment Simpson's 1/3rd Rule

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple segment Trapezoidal Rule, one can subdivide the interval

[a, b] into n segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval [a, b] into equal segments, hence the segment width

$$h = \frac{b-a}{n} \qquad \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

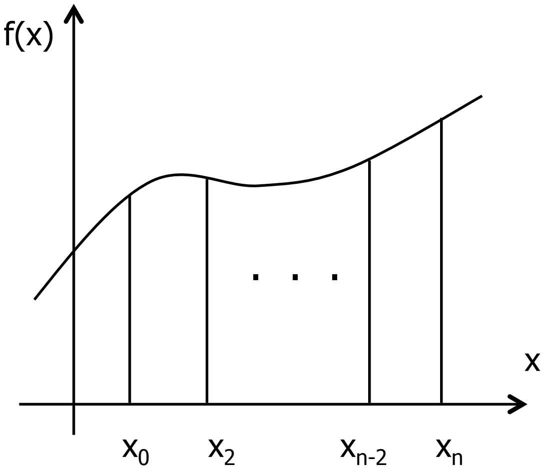
where

$$x_0 = a \qquad x_n = b$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots..$$

$$\ldots. + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + \ldots$$

$$+(x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

Multiple Segment Simpson's 1/3rd Rule

$$\ldots + (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + \ldots$$

$$+ (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h \qquad\qquad i = 2, 4, \ldots, n$$

Multiple Segment Simpson's 1/3rd Rule

Then

$$\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right]+\ldots$$

$$+2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Multiple Segment Simpson's 1/3rd Rule

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\left\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\right\} + \ldots\right]$$

$$\ldots + 2\left\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\right\} + f(x_n)\}]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$= \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

Example 2

Use 4-segment Simpson's 1/3rd Rule to approximate the distance

covered by a rocket from t= 8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use four segment Simpson's 1/3rd Rule to find the approximate value of x.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

a) Using n segment Simpson's 1/3rd Rule,

$$h = \frac{30-8}{4} = 5.5$$

So

$$f(t_0) = f(8)$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(t_4) = f(30)$$

Solution (cont.)

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]$$

Solution (cont.)

cont.

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)\right]$$

$$= \frac{11}{6}\left[177.2667 + 4(320.2469) + 4(676.0501) + 2(484.7455) + 901.6740\right]$$

$$= 11061.64\ m$$

Solution (cont.)

b) In this case, the true error is

$$E_t = 11061.34 - 11061.64 = -0.30\ m$$

c) The absolute relative true error

$$|\in_t| = \left|\frac{11061.34 - 11061.64}{11061.34}\right| \times 100\%$$

$$= 0.0027\%$$

Solution (cont.)

Table 1: Values of Simpson's 1/3rd Rule for Example 2 with multiple segments

n	Approximate Value	E_t	$\|\in_t\|$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Error in the Multiple Segment Simpson's 1/3rd Rule

The true error in a single application of Simpson's 1/3rd Rule is given as

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. The error in n segment Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1) = -\frac{h^5}{90} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2) = -\frac{h^5}{90} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

Error in the Multiple Segment Simpson's 1/3rd Rule

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i) = -\frac{h^5}{90} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

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$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{4}\left(\zeta_{\frac{n}{2}}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Error in the Multiple Segment Simpson's 1/3rd Rule

Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i \quad = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \qquad = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Error in the Multiple Segment Simpson's 1/3rd Rule

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of

$$f^{(4)}(x),\ a < x < b$$

Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

Integration

Topic: Gauss Quadrature Rule of Integration

Major: General Engineering

Two-Point Gaussian Quadrature Rule

Basis of the Gaussian Quadrature Rule

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \cong c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Basis of the Gaussian Quadrature Rule

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns x_1 and x_2. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Basis of the Gaussian Quadrature Rule

The four unknowns x_1, x_2, c_1 and c_2 are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

Basis of the Gaussian Quadrature Rule

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^{\,2} + a_3x_1^{\,3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\,2} + a_3x_2^{\,3}\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^{\,2} + a_3x_1^{\,3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\,2} + a_3x_2^{\,3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^{\,2} + c_2x_2^{\,2}\right) + a_3\left(c_1x_1^{\,3} + c_2x_2^{\,3}\right)$$

Basis of the Gaussian Quadrature Rule

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2$$

$$\frac{b^3 - a^3}{3} = c_1 x_1{}^2 + c_2 x_2{}^2$$

$$\frac{b^4 - a^4}{4} = c_1 x_1{}^3 + c_2 x_2{}^3$$

Basis of Gauss Quadrature

The previous four simultaneous nonlinear Equations have only one acceptable solution,

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2} \qquad x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$c_1 = \frac{b-a}{2} \qquad c_2 = \frac{b-a}{2}$$

Basis of Gauss Quadrature

Hence Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

Higher Point Gaussian Quadrature Formulas

Higher Point Gaussian Quadrature Formulas

$$\int_a^b f(x)dx \cong c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients c_1, c_2, and c_3, and the functional arguments x_1, x_2, and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right)dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point Gauss Quadrature Rule are given for integrals

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Table 1 (cont.) : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.2386191860 x_4 = 0.2386191860 x_5 = 0.661209386 x_6 = 0.932469514

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$? The answer lies in that any integral with limits of $\left[a,\, b\right]$ can be converted into an integral with limits $\left[-1,\, 1\right]$ Let

$$x = mt + c$$

If $x = a,$ then $t = -1$

If $x = b,$ then $t = 1$

Such that:

$$m = \frac{b-a}{2}$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Then $c = \frac{b+a}{2}$ Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \qquad dx = \frac{b-a}{2}dt$$

Substituting our values of x, and dx into the integral gives us

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx$$

Example 1

For an integral $\int_a^b f(x)dx,$ derive the one-point Gaussian Quadrature Rule.

Solution

The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$

Solution

Assuming the formula gives exact values for integrals

$$\int_{-1}^{1} 1dx, \quad \text{and} \quad \int_{-1}^{1} xdx,$$

$$\int_a^b 1dx = b - a = c_1 \qquad \int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1$$

Since $c_1 = b - a,$ the other equation becomes

$$(b - a)x_1 = \frac{b^2 - a^2}{2} \qquad x_1 = \frac{b + a}{2}$$

Solution (cont.)

Therefore, one-point Gauss Quadrature Rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{b+a}{2}\right)$$

Example 2

a) Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from t=8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

b) Find the true error, E_t for part (a).

c) Also, find the absolute relative true error, $|\varepsilon_a|$ for part (a).

Solution

First, change the limits of integration from [8,30] to [-1,1]

by previous relations as follows

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

Solution (cont)

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$$c_1 = 1.000000000$$

$$x_1 = -0.577350269$$

$$c_2 = 1.000000000$$

$$x_2 = 0.577350269$$

Solution (cont.)

Now we can use the Gauss Quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19)+11c_2 f(11x_2+19)$$

$$=11f(11(-0.5773503)+19)+11f(11(0.5773503)+19)$$

$$=11f(12.64915)+11f(25.35085)$$

$$=11(296.8317)+11(708.4811)$$

$$=11058.44\ m$$

Solution (cont)

since

$$f(12.64915) = 2000\,ln\left[\frac{140000}{140000 - 2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000\,ln\left[\frac{140000}{140000 - 2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

Solution (cont)

b) The true error, E_t , is

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061.34 - 11058.44$$

$$= 2.9000\ m$$

c) The absolute relative true error, $|\in_t|$, is (Exact value = 11061.34m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100\%$$

$$= 0.0262\%$$

Exercise 5 (Feb. 24)

Return Feb. 27, 9:15 a.m.

Free Training

- Write a program code for numerically computing a definite integral, using multiple segments (free parameters are: number of segments n, lower and upper bound a and b, step size $h = (b - a)/n$). Prepare the following three methods:

  1. Trapezium Rule
  2. Simpson 1/3 Rule
  3. Gaussian Two-Point Quadrature

  Test your programm for $f(x) = x$ ($a = 0$, $b = 2$) and $f(x) = x^2 - 3x$ ($a = -3$, $b = 6$)

Assignment for Afternoon/Home Work, 20 Points

- **Exercise 5.1, 5 points**: Trapezium Rule.
  Integrate numerically the definite integral

$$\int_0^2 (2 + \cos(2\sqrt{x}))dx \tag{0.1}$$

  using the Trapezium rule. Use $n = 2, 10, 100, 1000, 10000$, print the result.

- **Exercise 5.2, 5 points**: Simpson 1/3 rule.
  Integrate the definite integral of 5.1 using Simpson's 1/3 rule, for $n = 2, 10, 100, 1000, 10000$, print the results.

- **Exercise 5.3, 5 points**: Gaussian two point quadrature.
  Integrate the definite integral of 5.1 using the Gaussian two point quadrature, for $n = 2, 10, 100, 1000, 10000$ intervals of $[a, b]$, print the results.

- **Exercise 5.4, 5 points**: Accuracy and Errors.
  Evaluate the integral of 5.1 analytically. Compute the true error (absolute and relative) of the numerically computed integral for Trapezium, Simpson 1/3 rule and Gaussian two-point quadrature (for the $n = 2, 10, 100, 1000, 10000$) values. Put all results in a double logarithmic plot of error against n. What scaling of the error do you find?