Practical Numerical Training UKNum Linear Equations

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Programm:
1) Introduction
2) Gauss elimination
3) Gauss with partial pivoting
4) Determinants
5) LU decomposition
6) Inverse of matrix using LU

Linear Equations

[A][X] = [C]

Gaussian Elimination Gaussian Elimination with Partial Pivoting LU Decomposition / QR Decomposition **Gauss-Seidel Iteration** Sparce Matrix Solvers (e.g. Cholesky Decomposition)

Gaussian Elimination

Major: All Engineering Majors

http://numericalmethods.eng.usf.edu

Transforming Numerical Methods Education for STEM Undergraduates

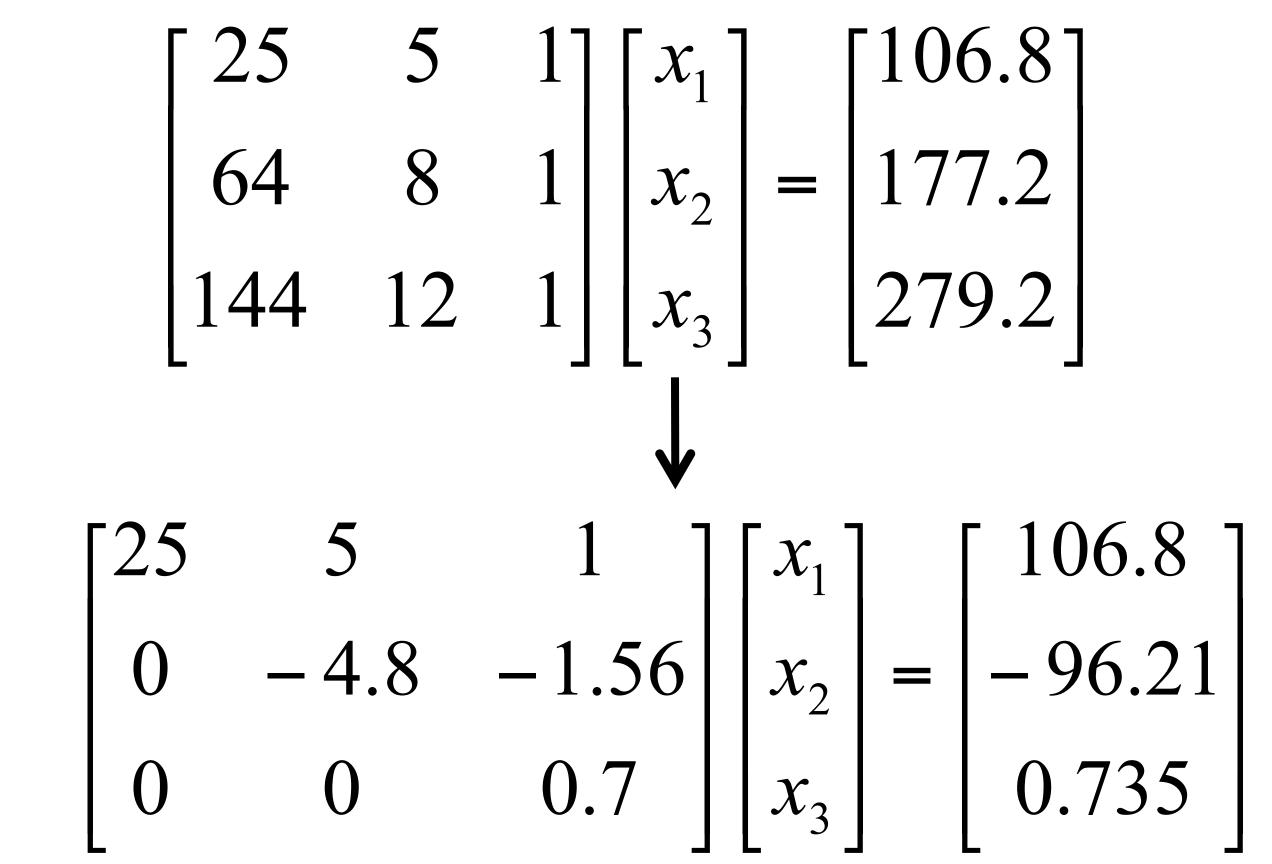
Author(s): Autar Kaw

Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps **1.** Forward Elimination 2. Back Substitution

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix



A set of n equations and n unknowns

 $a_{11}x_1 + a_{12}x_2 + a_{13}$

 $a_{21}x_1 + a_{22}x_2 + a_{23}$

 $a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3$

$$x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

$$x_{3} + \dots + a_{2n}x_{n} = b_{2}$$

$$x_{3} + \dots + a_{nn}x_{n} = b_{n}$$

(n-1) steps of forward elimination

Forward Elimination Step 1 For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21} .

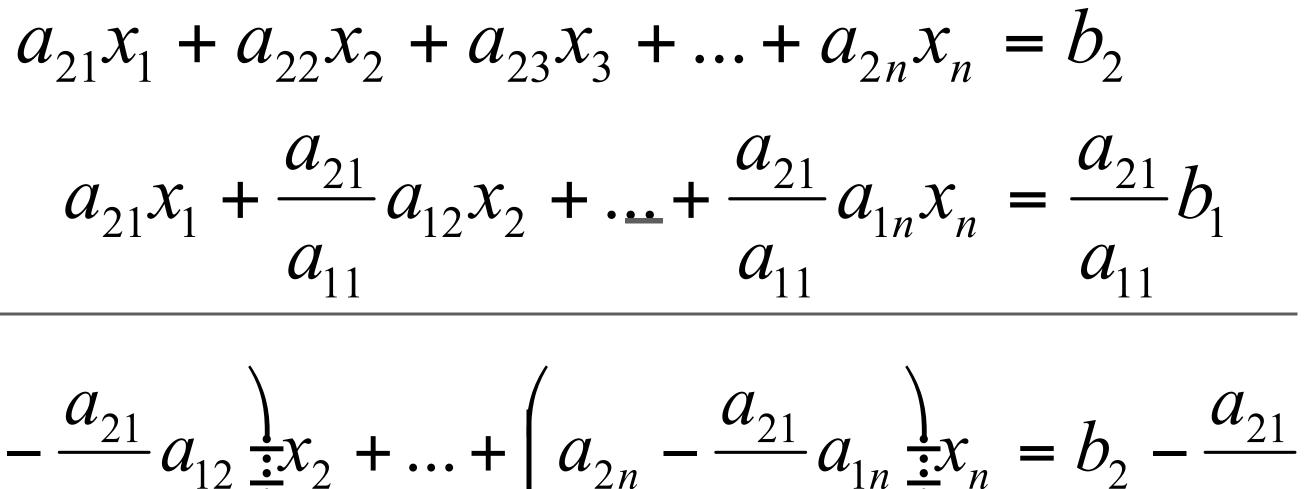
$$\begin{bmatrix} \frac{a_{21}}{a_{11}} \end{bmatrix} (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination Subtract the result from Equation 2.

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\frac{1}{x_2} + \dots + a_{11}\frac{1}{y_2}\right)$$

or $a'_{22}x_2 + \dots + a'_{2n}x_n = b'_2$



$$+\left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\frac{1}{\frac{1}{2}}x_n = b_2 - \frac{a_{21}}{a_{11}}b_1\right)$$

 $a_{11}x_1 + a_{12}x_2 + a_{13}$ $a'_{22}x_2 + a'_{23}$ $a'_{32}x_2 + a'_{33}$ $a_{n2}x_{2} + a_{n}$

End of Step 1

Repeat this procedure for the remaining equations to reduce the set of equations as

$$x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

$$x_{3} + \dots + a_{2n}x_{n} = b_{2}'$$

$$x_{3} + \dots + a_{3n}x_{n} = b_{3}'$$

$$a_{3}x_{3} + \dots + a_{nn}x_{n} = b_{n}'$$

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3$

 $a_{22}x_2 + a_2$

 \mathcal{A}



Step 2 Repeat the same procedure for the 3rd term of Equation 3.

$$x_{13}x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

$$y_{23}x_{3} + \dots + a_{2n}x_{n} = b_{2}'$$

$$y_{33}x_{3} + \dots + a_{3n}x_{n} = b_{3}''$$

$$a_{n3}^{"}x_3 + \dots + a_{nn}^{"}x_n = b_n^{"}$$

End of Step 2

- $a_{11}x_1 + a_{12}x_2 + a_{12}x_2$ $a'_{22}x_2 + a$
 - \mathcal{A}



At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$a_{13}x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

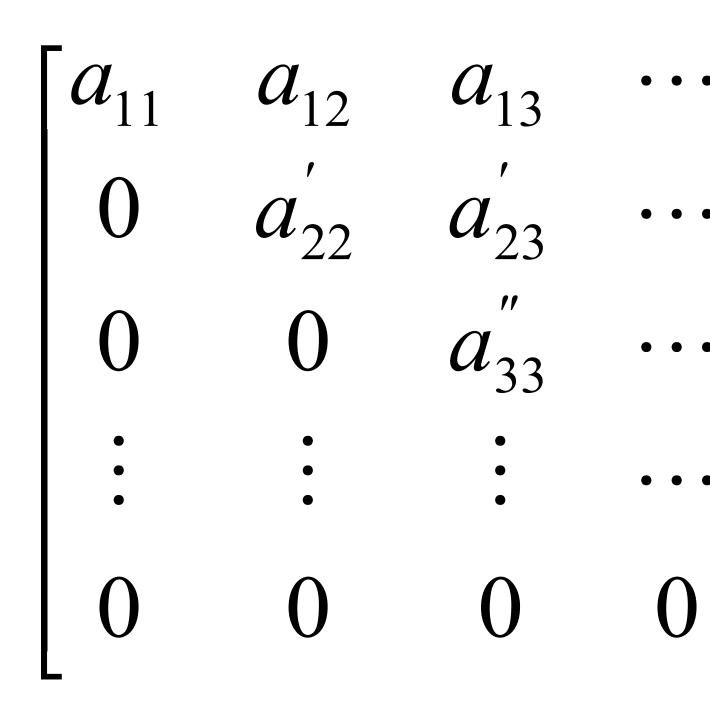
$$a_{23}x_{3} + \dots + a_{2n}x_{n} = b_{2}$$

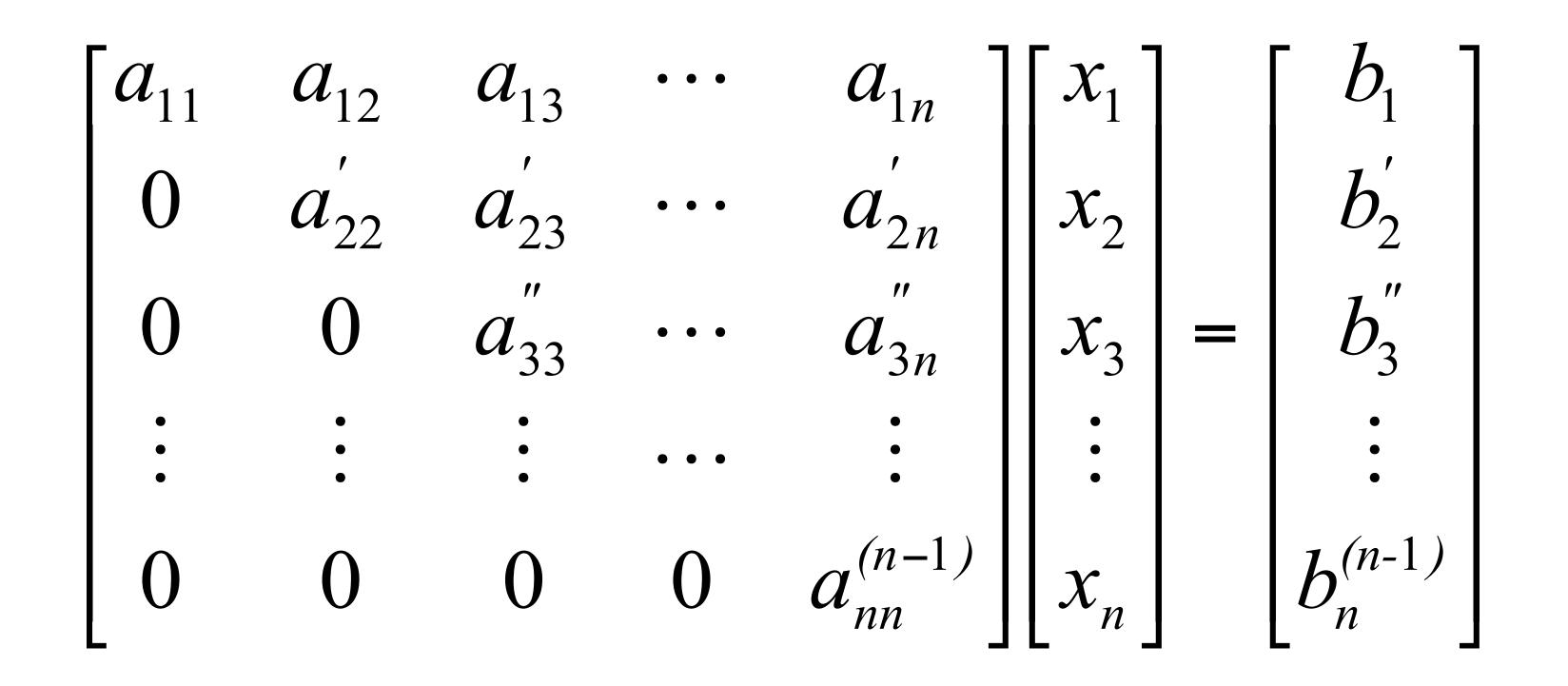
$$a_{33}x_{3} + \dots + a_{3n}x_{n} = b_{3}$$

$$a_{nn}^{(n-1)}x_{n} = b_{n}^{(n-1)}$$

End of Step (n-1)

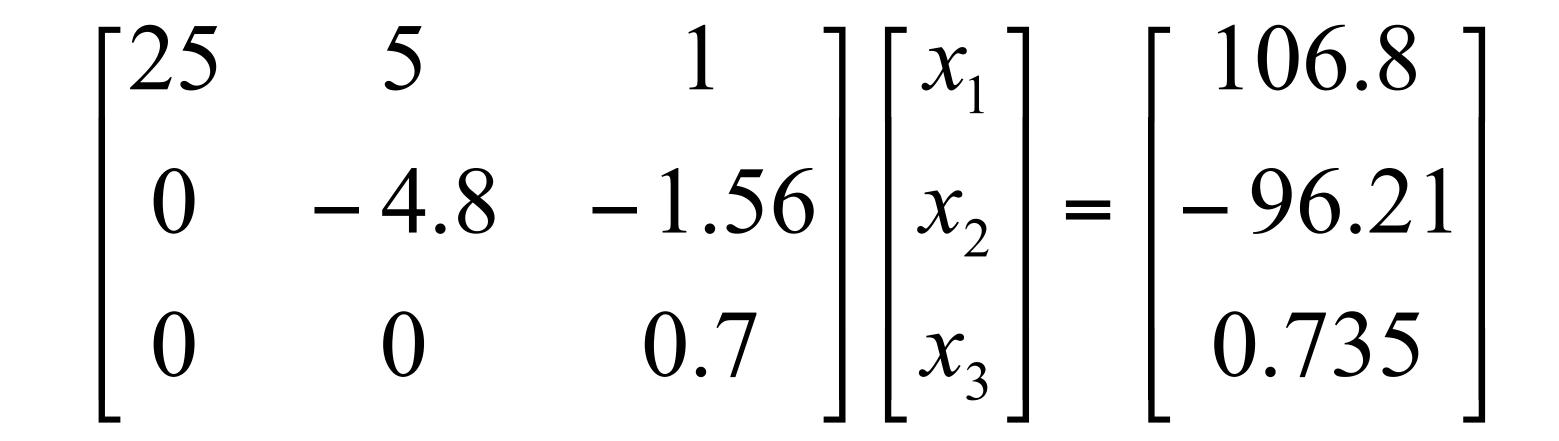
Matrix Form at End of Forward Elimination





Back Substitution

Solve each equation starting from the last equation



Example of a system of 3 equations

Back Substitution Starting Eqns

 $a_{11}x_1 + a_{12}x_2 +$

 $a'_{22}x_2 +$

$$a_{13}x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

$$a_{23}x_{3} + \dots + a_{2n}x_{n} = b_{2}$$

$$a_{33}x_{3} + \dots + a_{n}x_{n} = b_{3}^{"}$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

. .

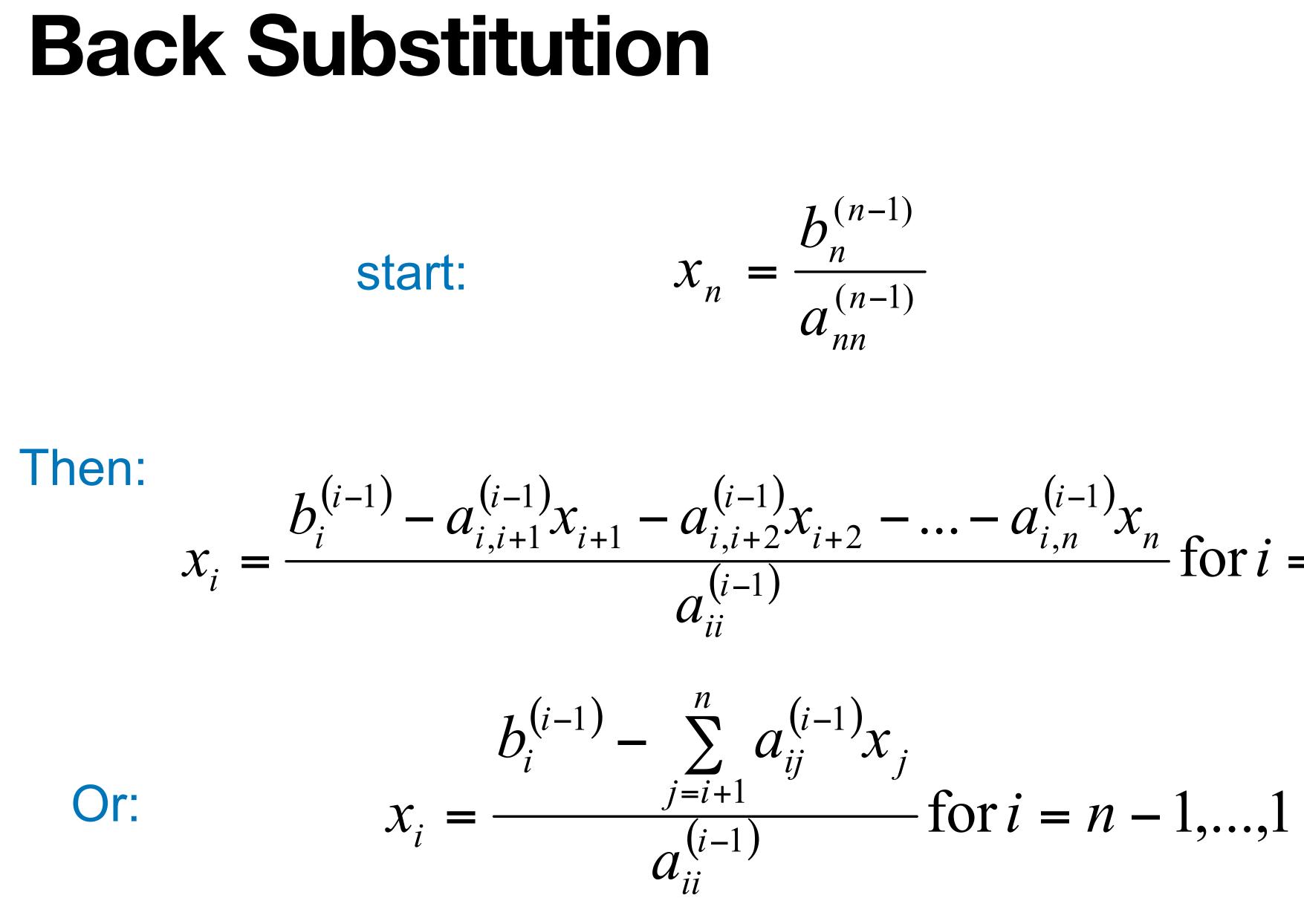
Back Substitution

Start with the last equation because it has only one unknown:

 $X_n =$

$$\frac{b_n^{(n-1)}}{a^{(n-1)}}$$

nn



$$x_{i+2} - \dots - a_{i,n}^{(i-1)} x_n$$
 for $i = n - 1, \dots, 1$

Naïve Gauss Elimination Pitfalls

<u>http://</u> numericalmethods.eng.usf.edu

Pitfall#1. Division by zero

Example 1:

 $10x_{2} - 7x_{3} = 3$

 $\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$

 $6x_1 + 2x_2 + 3x_3 = 11$ $5x_1 - x_2 + 5x_3 = 9$

 $5 -1 5 | x_3 | 9 |$

Is division by zero an issue here? **Example 2**:

- $12x_1 + 10x_2 7x_3 = 15$ $6x_1 + 5x_2 + 3x_3 = 14$ $5x_1 - x_2 + 5x_3 = 9$

- $\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$

Is division by zero an issue here? YES **Example 2**:

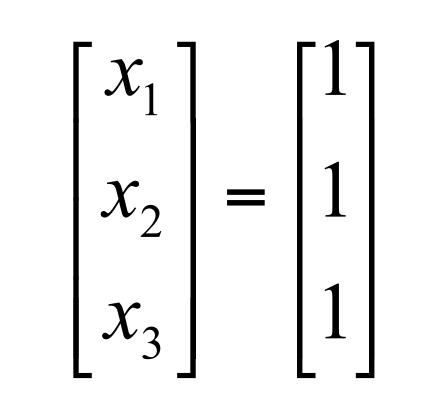
- $12x_1 + 10x_2 7x_3 = 15$
- $6x_1 + 5x_2 + 3x_3 = 14$
- $24x_1 x_2 + 5x_3 = 28$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix}$$

Division by zero is a possibility at any step of forward elimination

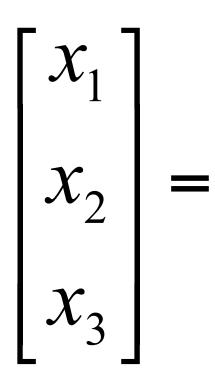
Pitfall#2. Large Round-off Errors

Example:



- $\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$
 - **Exact Solution**

Pitfall#2. Large Round-off Errors



 $\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$

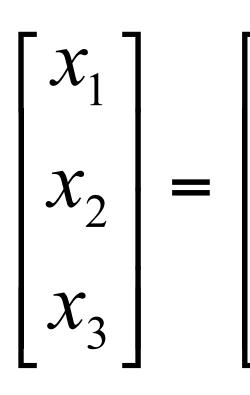
Solve it on a computer using 6 significant digits with chopping

 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \end{bmatrix}$ $|x_3| = 0.999995|$

Pitfall#2. Large Round-off Errors

$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$

Solve it on a computer using 5 significant digits with chopping



 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$

Is there a way to reduce the round off error?

Avoiding Pitfalls

- Does decrease round-off error
- Does not avoid division by zero

Increase the number of significant digits

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

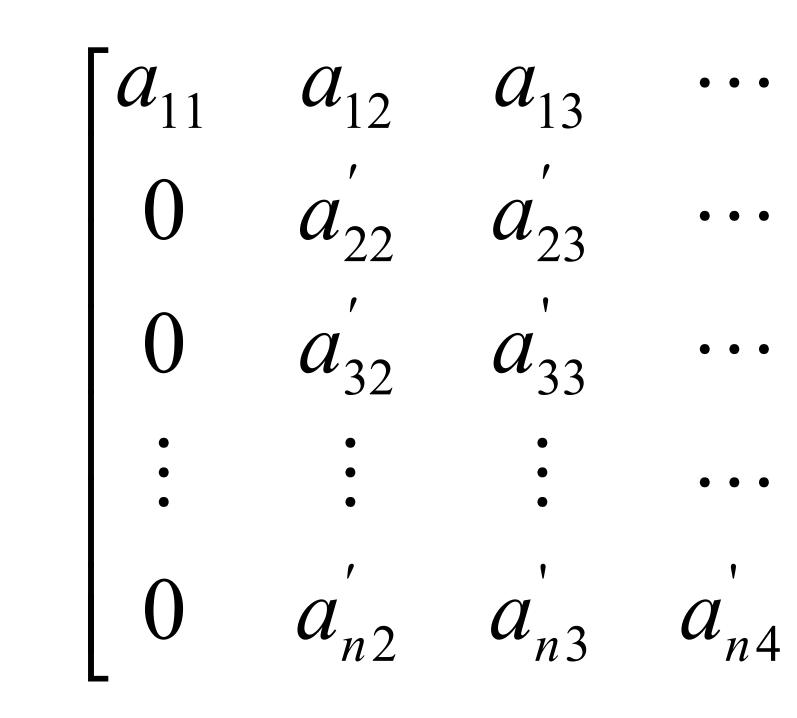
What is Different About Partial Pivoting?

$$|a_{kk}|, |a_{k+1,k}|, \dots, |a_{nk}|$$

At the beginning of the kth step of forward elimination, find the maximum of

If the maximum of the values is $|a_{pk}|$ in the p th row, $k \le p \le n$, then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

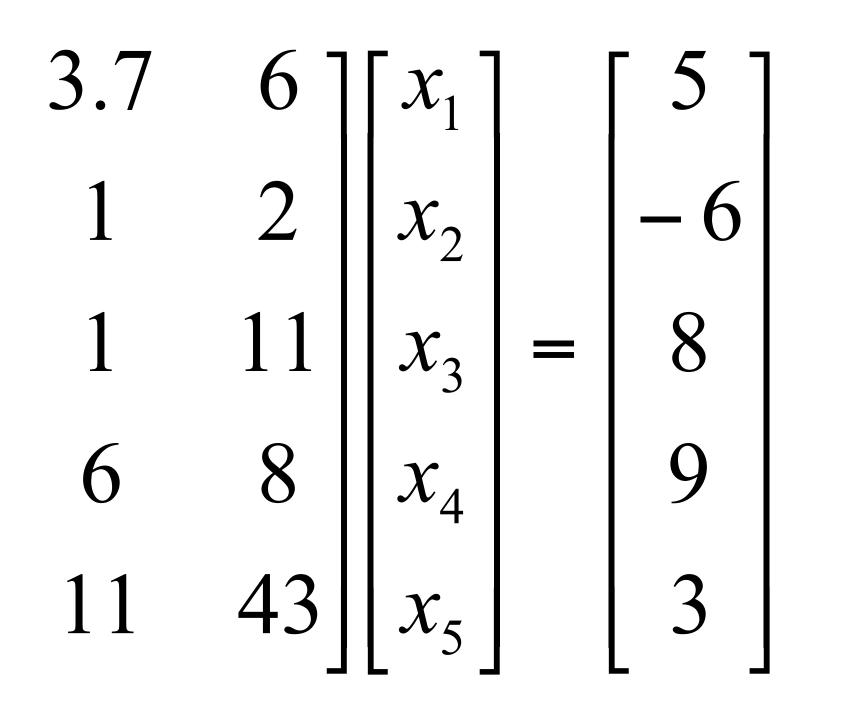


$$\begin{array}{ccc} & & & a_{1n} \\ & & & a_{2n} \\ & & & a_{3n} \\ & & & \vdots \\ & & & a_{nn} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_3 \\ \vdots \\ & & & \\ \vdots \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ &$$

Example (2nd step of FE)

[6	14	5.1
0	-7	6
0	4	12
0	9	23
0	-17	12

Which two rows would you switch?



Example (2nd step of FE)

$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$ Switched Rows

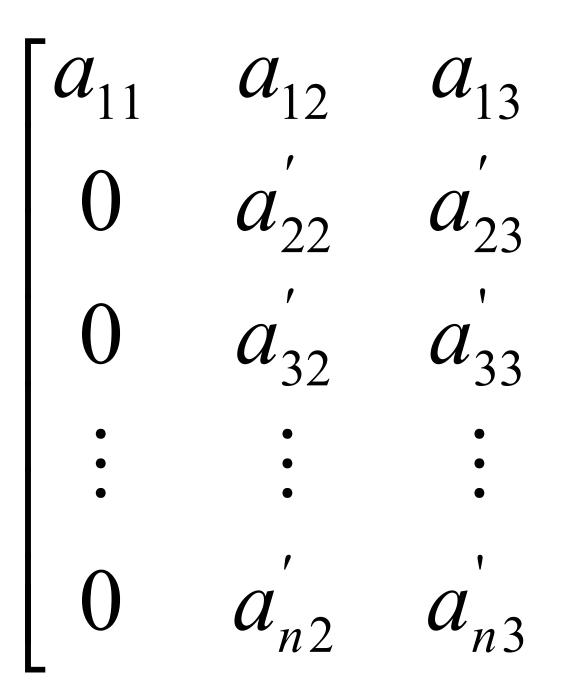
Gaussian Elimination with Partial Pivoting

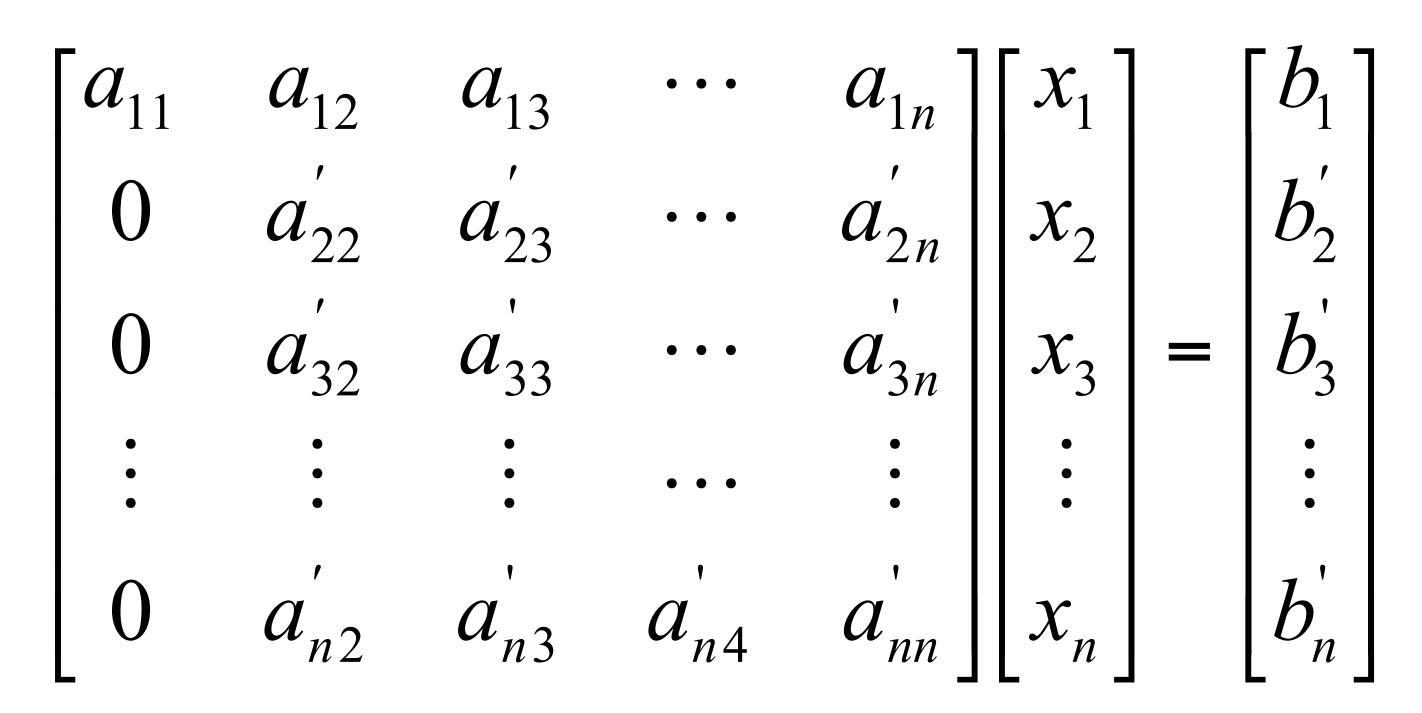
A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps **1.** Forward Elimination 2. Back Substitution

Same as naïve Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

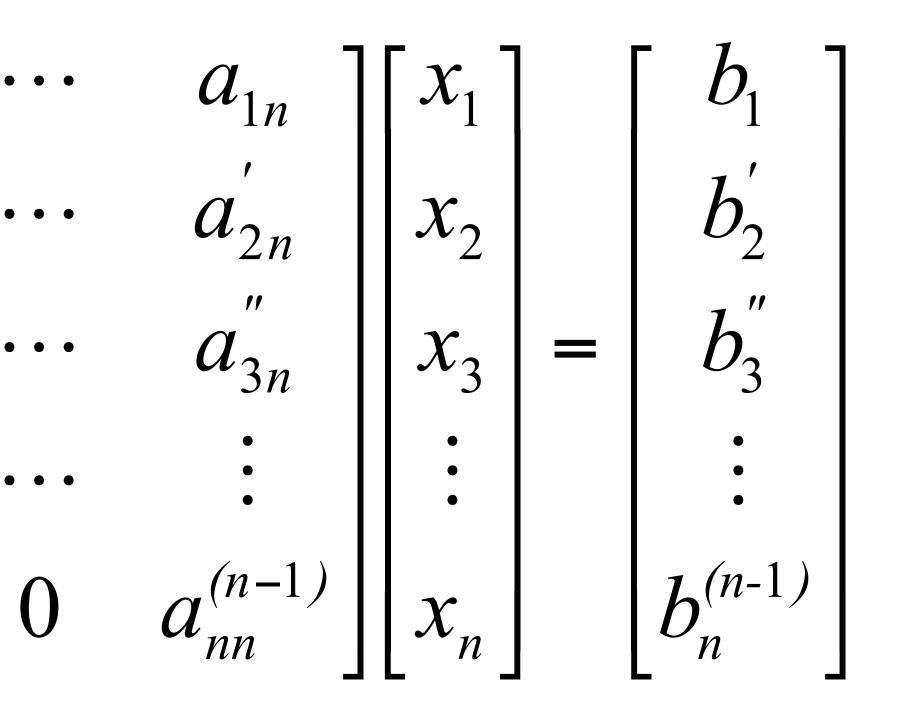
Example: Matrix Form at Beginning of 2nd Step of Forward Elimination





Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ 0 & a_{22}' & a_{23}' & \cdots \\ 0 & 0 & a_{33}'' & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Back Substitution Starting Eqns

 $a_{11}x_1 + a_{12}x_2 +$

 $a'_{22}x_2 +$

$$a_{13}x_{3} + \dots + a_{1n}x_{n} = b_{1}$$

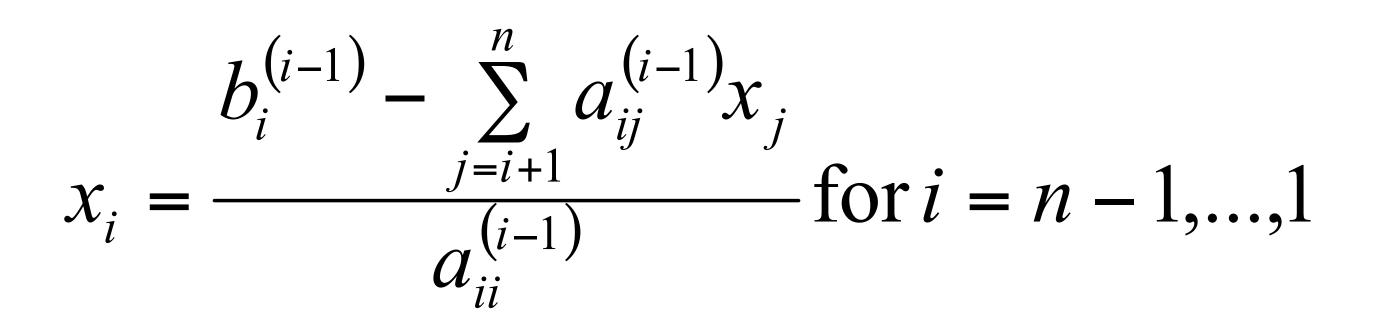
$$a_{23}x_{3} + \dots + a_{2n}x_{n} = b_{2}$$

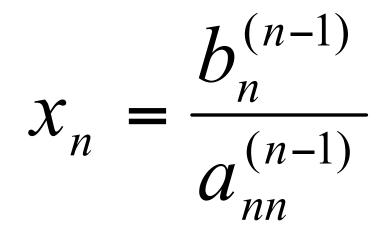
$$a_{33}x_{3} + \dots + a_{n}x_{n} = b_{3}^{"}$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

. .

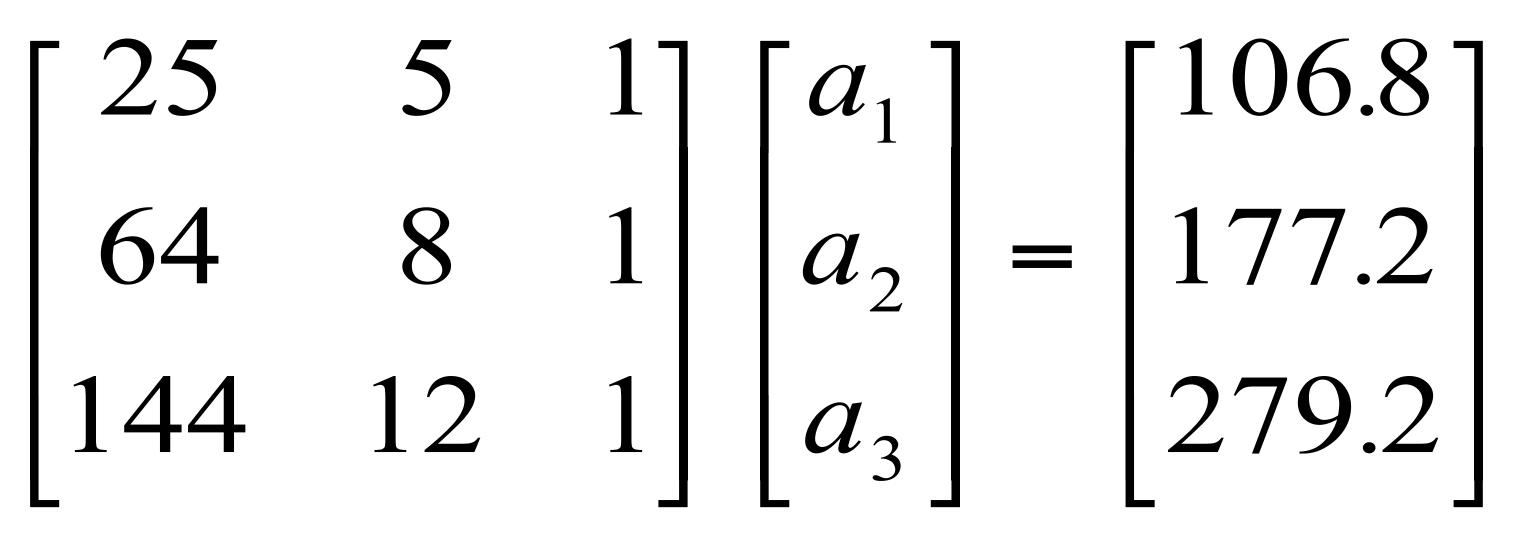
Back Substitution ...all the same!







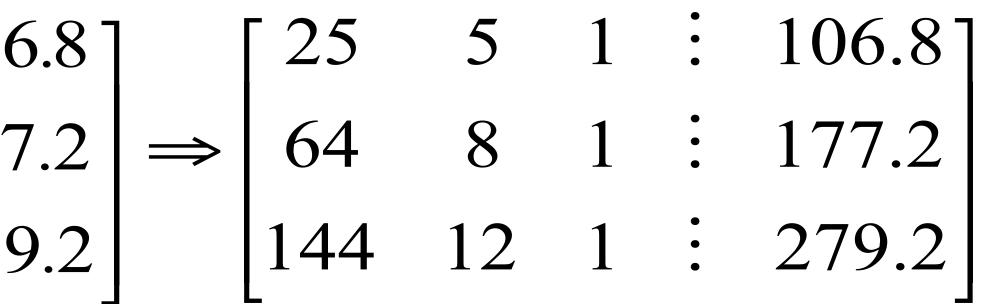
Solve the following set of equations by Gaussian elimination with partial pivoting



Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106 \\ 177 \\ 279 \end{bmatrix}$$

Forward Elimination
 Back Substitution



Number of Steps of Forward Elimination

Number of steps of forward elimination is (n-1)=(3-1)=2

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below: |25|, |64|, |144|
 - Largest absolute value is 144 and exists in row 3. • Switch row 1 and row 3.

 - $\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$

$$\left| \begin{array}{cccccccc} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{array} \right|$$

Forward Elimination: Step 1 (cont.)

 $\begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ -\begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix}$ $\begin{bmatrix} 0 & 2.667 & 0.5556 \\ \vdots & 53.10 \end{bmatrix}$

Subtract the result from Equation 2

Substitute new equation for Equation 2

 $\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$ Divide Equation 1 by 144 and multiply it by 64, $\frac{64}{144} = 0.4444$

 $\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$ 5 1 : 106.8 25

Forward Elimination: Step 1 (cont.)

[144		2		1	• • •	279
0	2.6	667	().5556	• •	53.1
25	12 2.667 5			1	• •	106
144	12	1	• •	279.2	$\geq 0.$	1736

Subtract the result from Equation 3

Substitute new equation for Equation 3

Divide Equation 1 by 144 and multiply it by 25, $\frac{25}{144} = 0.1736$

 $6 = \begin{bmatrix} 25.00 & 2.083 & 0.1736 \\ \vdots & 48.47 \end{bmatrix}$

 $\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$ - $\begin{bmatrix} 25 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix}$

[0 2.917 0.8264 : 58.33]

 $\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$

Forward Elimination: Step 2

- Examine absolute values of second column, second row 2.667, 2.917 and below.

 - Switch row 2 and row 3.

[144	12	1	• •	279.2]	[144	12	1	• •	279.2]
	0	2.667	0.5556	• •	53.10 ⇒	> 0	2.917	0.8264	• •	58.33
	0	2.917	0.8264	• •	58.33	0	2.667	0.5556	• •	53.10

Largest absolute value is 2.917 and exists in row 3.

Forward Elimination: Step 2 (cont.)

 $\begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \times 0.9143 = \begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix}$ $\begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \\ -\begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$ Equation 3

Subtract the result from

Substitute new equation for Equation 3

 $\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$ Divide Equation 2 by 2.917 and multiply it by 2.667, $\frac{2.667}{2.917} = 0.9143.$

[144 12 1 **:** 279.2] 0.8264 : 58.33 2.917 0 -0.2 : -0.230 0



Back Substitution

 $\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow$

 $-0.2a_3 = -0.23$ $a_3 = \frac{-0.23}{-0.2}$ = 1.15

$$\Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for *a*₃

Back Substitution (cont.)

 $\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.82 \\ 0 & 0 & -($

 $2.917a_2 + 0.8264a_3 = 58.33$

$$\begin{bmatrix} 1 \\ 8264 \\ -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for *a*₂

 $a_2 = \frac{58.33 - 0.8264a_3}{2.917}$

 $58.33 - 0.8264 \times 1.15$

2.917

= 19.67

Back Substitution (cont.)

 $144a_1 + 12a_2 + a_3 = 279.2$ $a_1 = \frac{279.2 - 12a_2 - a_3}{144}$ $279.2 - 12 \times 19.67 - 1.15$ 144

 $\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$ Solving for *a*₁

= 0.2917

Gaussian Elimination with Partial Pivoting Solution

$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$

 $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \end{bmatrix}$ *a*₃ 1.15



Determinant of a Square Matrix

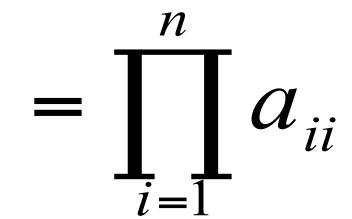
Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then det(A)=det(B)

Theorem of Determinants

The determinant of an upper triangular matrix [A]_{nxn} is given by

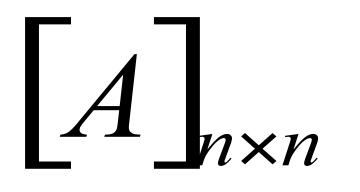
$$\det(\mathbf{A}) = a_{11} \times a_2$$

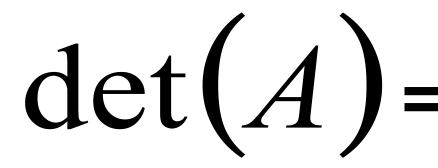


 $_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.





$A_{n \times n} \longrightarrow U_{n \times n}$

det(A) = det(U)

Example

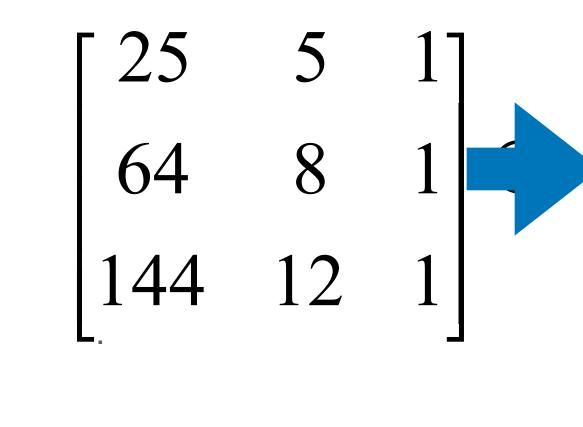
Using naïve Gaussian elimination find the determinant of the following square matrix.



- $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$
- Forward elimination...

Finding the Determinant

After forward elimination



$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$det(A) = u_{11} \times u_{22} \times u_{33}$$

= 25 × (-4.8)×0.7
= -84.00

Determinants and Pivoting?

What to do if division by zero may occur?

You can do pivoting, but be careful! The determinant may change sign!

Theorem 3

 $[A]_{nxn}$ is a n x n matrix. If $[B]_{nxn}$ is a matrix created by switching collumns or rows in $[A]_{nxn}$, then det(A)=-det(B).



Major: All Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates

02/15/10

http://numericalmethods.eng.usf.edu

Authors: Autar Kaw

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition is another method to solve a set of simultaneous linear equations

Method

- For most non-singular matrix [A] that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as
 - [A] = [L][U]
 - where
 - [L] = lower triangular matrix
 - [U] = upper triangular matrix

How does LU Decomposition work?

- If solving a set of linear equations
- If [A] = [L][U] then Multiply by Which gives

- Remember $[L]^{-1}[L] = [I]$ which leads to
 - Now, if [I][U] = [U] then
 - Now, let
 - Which ends with [L][Z] = [C] (1) $[U][X] = [Z] \quad (2)$

- [A][X] = [C]
- [L][U][X] = [C]
- $[L]^{-1}$
 - $[L]^{-1}[L][U][X] = [L]^{-1}[C]$
- $[I][U][X] = [L]^{-1}[C]$
- $[U][X] = [L]^{-1}[C]$
- $[L]^{-1}[C] = [Z]$
- and

Given [A][X] = [C]**1.Decompose** [A] into [L] and [U]**2.Solve** [L][Z] = [C] for [Z]

3.Solve [U][X] = [Z] for [X]

How can this be used?

When is LU Decomposition better than Gaussian Elimination?

To solve [A][X] = [B]

Table. Time taken by methods

Gaussian Elimination

$$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)\frac{1}{\frac{1}{5}}$$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

LU Decomposition

$$T\left(\frac{8n^{3}}{3} + 12n^{2} + \frac{4n}{3}\frac{1}{j}\right)$$



To find inverse of [A]

Time taken by Gaussian Elimination

$$= n \left(CT \mid_{FE} + CT \mid_{BS} \right)$$
$$= T \left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3} \right) \frac{1}{5}$$

 Table 1 Comparing computational times of finding inverse of a matrix using

 LU decomposition and Gaussian elimination.

n 1

$$CT|_{inverse GE} / CT|_{inverse LU}$$
 3.

Time taken by LU Decomposition

$$= CT \mid_{LU} + n \times CT \mid_{FS} + n \times CT \mid_{BS}$$
$$= T \left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3} \frac{1}{\dot{j}} \right)$$

10	100	1000	10000
3.28	25.83	250.8	2501

Method: [A] Decompose to [L] and [U]

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

 $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$ $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 144 & 12 & 1 \end{bmatrix}$ Step 1: $\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & -16.8 & -4.76 \end{bmatrix}$

Using the Forward Elimination Procedure of Gauss Elimination

Finding the [U] Matrix

Matrix after Step 1:

Step 2:
$$\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

0



$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

 $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \end{bmatrix}$ 0 0.7

Finding the [L] matrix

From the first step elimination of forward 25 5 64 8 144 12

 $\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$

Using the multipliers used during the Forward Elimination Procedure

1

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

1
 $\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$

Finding the [L] Matrix

From the second step of forward elimination

tep of forward elimination $\begin{bmatrix} 25 & 5\\ 0 & -4.8\\ 0 & -16.8 \end{bmatrix}$

 $\begin{bmatrix} L \end{bmatrix} = \begin{bmatrix} 1 \\ 2.56 \\ 5.76 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -1.56 \\ -4.76 \end{bmatrix} \quad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$\begin{bmatrix} L \end{bmatrix} U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = ?$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

Using the procedure for finding the [L] and [U] matrices

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$



Set [L][Z] = [C]

Solve for [Z]

 $z_1 = 106.8$ 2.56z₁ + z₂ = 177.2 5.76z₁ + 3.5z₂ + z₃ = 279.2

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example

Complete the forward substitution to solve for [Z]

$$z_{1} = 106.8$$

$$z_{2} = 177.2 - 2.56z_{1}$$

$$= 177.2 - 2.56(106.8)$$

$$= -96.2$$

$$z_{3} = 279.2 - 5.76z_{1} - 3.5z_{2}$$

$$= 279.2 - 5.76(106.8) - 3.5(-3.5)$$

$\begin{bmatrix} Z \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$ (-96.21)



Set [U][X] = [Z] 0

-250

Solve for [X]

The 3 equations become

 $25a_1 + 5a_2 + a_3 = 106.8$

$$\begin{bmatrix} 5 & 1 \\ -4.8 & -1.56 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

 $-4.8a_2 - 1.56a_3 = -96.21$ $0.7a_3 = 0.735$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$
$$a_3 = \frac{0.735}{0.7}$$
$$a_3 = 1.050$$

Substituting in a₃ and using the second equation

$$-4.8a_{2} - 1.56a_{3} = -96.21$$

$$a_{2} = \frac{-96.21 + 1.56a_{3}}{-4.8}$$

$$a_{2} = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_{2} = 19.70$$

Example

Substituting in a₃ and a₂ using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_{1} = \frac{106.8 - 5a_{2} - a_{3}}{25}$$
$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$
$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Finding the inverse of a square matrix

The inverse [B] of a square matrix [A] is defined as

[A][B] = [I] = [B][A]

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse? Assume the first column of [B] to be $[b_{11} \ b_{12} \ \dots \ b_{n1}]^T$ Using this and the definition of matrix multiplication

First column of [*B*]

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [B] can be found in the same manner

Second column of [B]

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

- Find the inverse of a square matrix [A]
 - $\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$
- Using the decomposition procedure, the [L] and [U] matrices are found to be

Solving for the each column of [B] requires two steps

1)Solve [L] [Z] = [C] for [Z]

2)Solve [U] [X] = [Z] for [X]

 $[L][Z] = [C] \rightarrow$ Step 1:

This generates the equations:

- 2.56z
- $5.76z_1 + 3.5z_2 + z_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

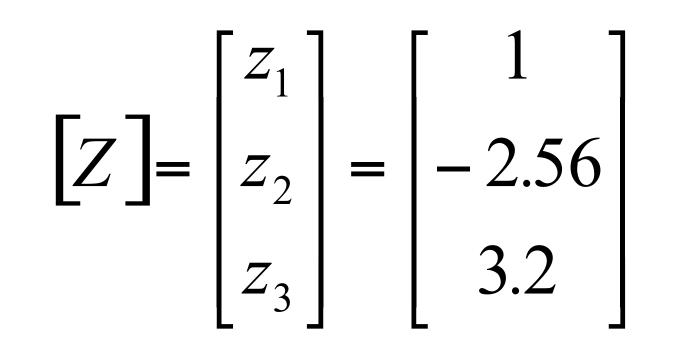
$$z_{1} = 1$$

$$z_{1} + z_{2} = 0$$

$$z_{1} + z_{2} = 0$$

Solving for [Z]

 $z_{1} = 1$ $z_{2} = 0 - 2.56z_{1}$ = 0 - 2.56(1) = -2.56 $z_{3} = 0 - 5.76z_{1} - 3.5z_{2}$ = 0 - 5.76(1) - 3.5(-2.56) = 3.2



Solving [U][X] = [Z] for [X]

 $25b_{11} + 5b_{21} + b_{31} = 1$ $-4.8b_{21} - 1.56b_{31} = -2.56$ $0.7b_{31} = 3.2$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -\frac{-4.8}{25}$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.$$

So the first column of the inverse of [A] is:

-0.9524

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

.04762

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix}^{1} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$[A][A]^{-1} =$$

- The inverse of [A] is
 - -0.083330.03571] 1.417 -0.4643 -5.000 1.429
- To check your work do the following operation
 - $[I] = [A]^{-1}[A]$

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/gauss_seidel.html

Free Training

- Start writing routines using the
 - 1. Naive Gaussian Elimination Method and
 - 2. Naive Gaussian Elimination with Partial Pivoting.

to solve the following set of linear equations:

where

$$\mathbf{A} = \begin{pmatrix} 13 & 4 & 7 & 9\\ 10 & 6 & 5 & 12\\ 1 & 8 & 2 & 16\\ 3 & 14 & 15 & 11 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 111\\ 118\\ 114\\ 163 \end{pmatrix}$$

Assignment for the Afternoon / Homework, 20 Points

- method to calculate the determinant of matrix A.
- square matrix. Use the above matrix A to test your program.
 - Print out the L and U parts of the matrix \mathbf{A} .
 - Solve the above equation.
 - Optional: Compute the inverse of the matrix A.

Exercise:

3. Think how to implement the LU decomposition of a matrix A

 $\mathbf{A} \cdot \mathbf{x} = \mathbf{b},$

• Exercise 1, 10 points: Print the upper triangular matrix and the new b-vector after the forward Gaussian elimination and show the solution of \mathbf{x} . Also, use the

• Exercise 2, 10 points: Write a routine which does the LU decomposition on a