**Numerisches Praktikum – Numerical Practical Training**

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# Ordinary Differential Equations

**Return by 9:15 a.m. Feb 22nd**
**as `.pdf` by Mail to:** `cecil@mpia.de`

## Free Training

- Write a program code for solving a system of 1st order ordinary differential equations of the type $\mathbf{y}' \equiv d\mathbf{y}/dx = f(\mathbf{y}, x)$ using the
  1. Euler forward method
  2. Runge-Kutta 2nd order (RK2): i.e. Midpoint Rule:

$$k_1 = hf\left(x_n, y_n\right) \tag{1}$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \tag{2}$$

$$y_{n+1} = y_n + k_2 + O(h^3) \tag{3}$$

  with a constant integration step $h$, as presented in the lecture.

## Assignment for the Afternoon / Homework

- **Exercise.1, 8 points**: Accuracy and precision.
  Solve the differential equation $y' = \exp(x)$ using both the Euler and the RK2 scheme. Start with the initial values $x = 0$ and $y(0) = 1$ and integrate up to $x = 1$ with constant integration steps $h = 1, 1/2, 1/3 \ldots 1/10$. Plot $y_i$ as a function of $x_i$ and compare the results directly with the analytical solution. Plot the absolute true error for $x = 1$ as a function of $h$ in a double logarithmic plot. Confirm the order of the integration scheme.

- **Exercise 2, 6 points**: Harmonic oscillator.
  Solve the $2^{\text{nd}}$ order ordinary differential equation (ODE) of a harmonic oscillator:

$$\ddot{x} \equiv \frac{d^2x}{d\tau^2} = -kx, \tag{4}$$

with $k = 16$. For this, first transform the $2^{\text{nd}}$ order ODE to a set of two $1^{\text{st}}$ order ODE's as shown in the lecture, i.e.:

$$\begin{aligned}
k_{11} &= \mathrm{dt} f\,(t_n, v_n) && (5)\\
k_{12} &= \mathrm{dt} f\,(t_n, x_n) && (6)\\
k_{21} &= \mathrm{dt} f\left(t_n + \frac{1}{2}dt, v_n + \frac{1}{2}k_{12}, ...\right) && (7)\\
k_{22} &= \mathrm{dt} f\left(t_n + \frac{1}{2}dt, x_n + \frac{1}{2}k_{11}, ...\right) && (8)\\
x_{n+1} &= x_n + k_{21} + O(\mathrm{dt}^3) && (9)\\
v_{n+1} &= v_n + k_{22} + O(\mathrm{dt}^3) && (10)
\end{aligned}$$

Start with the initial values $\tau = 0$, $x(0) = 0.8$ and $\dot{x}(0) = 0$ and integrate the system for several periods using the RK2. Find an integration step dt that conserves the energy $E = \frac{1}{2}kA^2$, where $A$ is the amplitude of the oscillation, with a relative true error of less than 0.01 at the end of the integration. Plot the results of $x(\tau)$ and $\dot{x}(\tau)$. Using the same integration step, how does the solution change when switching to the Euler scheme?

- **Exercise 3, 6 points**: Volterra-Lotka System
  Solve with your RK2 program the dimensonless coupled differential equation of the Volterra-Lotka system (see lecture):

$$\begin{aligned}
\dot{u}_1 &= u_1(1 - u_2)\\
\dot{u}_2 &= \alpha\, u_2(u_1 - 1)
\end{aligned}$$

with $\alpha = 0.5$, $u_1(0) = 1$ and $u_2(0) = 2, 3, 4$ and 5 from $\tau = 0$ to (at least) $\tau = 50$. Plot the solution of $u_1(\tau)$ and $u_2(\tau)$, e.g., for $u_2(0) = 3$. Find an appropriate timestep $h$ for the integration. Is there a conserved quantity of the dynamical system? Also plot $u_2(\tau)$ versus $u_1(\tau)$.