SOME LINEAR RECURRENCES WITH CONSTANT COEFFICIENTS

RICHARD J. MATHAR

Abstract. Linear homogeneous s-order recurrences with constant coefficients of the form $a(n) = d_1 a(n-1) + d_2 a(n-2) + \cdots + d_s a(n-s)$, $n \ge n_r$, have generating functions $A(x)$,

$$A(x) = \sum_{i=0} a(i)x^i = \frac{\sum_{n=0}^{n_r-1} a(n)x^n - \sum_{i=1}^{s} d_i x^i \sum_{n=0}^{n_r-i-1} a(n)x^n}{1 - \sum_{i=1}^{s} d_i x^i},$$

rational functions in x, where $a(0)$, $a(1)$ up to $a(r)$, $s \le r+1$, are a set of the first few coefficients of the Taylor series which are set up independently.

1. Introduction

1.1. **Aim and Notation.** We consider the (ordinary) generating functions $A(x)$ of sequences $a(n)$ of numbers indexed by $n = 0, 1, \ldots$, which are the expansion coefficients of the Taylor series [5]

$$A(x) = \sum_{i=0}^{\infty} a(i)x^i. \tag{1}$$

The generating functions are shown normalized in the sense that the first power of the Taylor expansion is the constant one; other offsets are essentially obtained by multiplication of the generating function with powers of x to shift the index up by arbitrary amounts.

Sequences with period length p after some optional non-periodic lower indices, $a(n) = a(n-p)$, or sequences with period length p and a half-period symmetry (odd symmetry in the speak of Fourier Transforms), $a(n) = -a(n-p/2)$, are just special cases of these recurrences, with values $|d_i|$ equal to one or zero.

Generating functions $A(x)$ with recurrences of constant coefficients and the restricted formats for inhomogeneities considered here are rational functions of x. Decompositions in partial fractions my help to write down the generating functions as sums of two or more other generating functions, which in turn means that a sequence may be a term-by-term sum of more "primitive" sequences that may be investigated by some kind of reverse engineering. (In these cases, `PURRS` [2] may propose closed-form expressions for $a(n)$.)

Sections 2 and 3 are explicit evaluations of the formula in the abstract for some simple cases; their limiting ratio are obtained from the characteristic function [10, 9]. Section 4 looks at the simplest forms of inhomogenous recurrences.

All of this is well known, and embodied by the `gfun` Maple package, for example [5, 14, 16]. My complementary Maple functions on this theme are available in http://www.mpia.de/~mathar/progs/GenFLinRec.mp.

Date: June 19, 2014.

1.2. **Generic Formula.** The generating function of recurrences [13]

$$a(n) = \sum_{i=1}^{s} d_i a(n-i) + b(n), \quad n \ge n_r, \tag{2}$$

are essentially the generating function of the homogeneous case ($b = 0$) plus the generating function $B(x) \equiv \sum_{n=0}^{\infty} b(n)x^n$ of the inhomogeneity alone [12]:

$$\begin{aligned}
A(x) &\equiv \sum_{n=0}^{\infty} a(n)x^n = \sum_{n=0}^{n_r-1} a(n)x^n + \sum_{n=n_r}^{\infty} a(n)x^n \\
&= \sum_{n=0}^{n_r-1} a(n)x^n + \sum_{n=n_r}^{\infty} \sum_{i=1}^{s} d_i a(n-i)x^n + \sum_{n=n_r}^{\infty} b(n)x^n \\
&= \sum_{n=0}^{n_r-1} a(n)x^n + \sum_{i=1}^{s} \sum_{n=n_r-i}^{\infty} d_i x^i a(n)x^n + \sum_{n=0}^{\infty} b(n)x^n - \sum_{n=0}^{n_r-1} b(n)x^n \\
&= \sum_{n=0}^{n_r-1} [a(n)-b(n)]x^n + \sum_{i=1}^{s} d_i x^i \left(A(x) - \sum_{n=0}^{n_r-i-1} a(n)x^n \right) + B(x). \qquad (3)
\end{aligned}$$

Separating terms proportional to $A(x)$ and those not depending on $A(x)$ we get

$$(1 - \sum_{i=1}^{s} d_i x^i)A = B + \sum_{n=0}^{n_r-1} [a(n)-b(n)]x^n - \sum_{i=1}^{s} d_i x^i \sum_{n=0}^{n_r-i-1} a(n)x^n. \tag{4}$$

Dividing through $1 - \sum_i d_i x^i$ generalizes the equation in the abstract to nonzero B.

2. 1-Term Homogeneous

Ordered according to increasing distance (stride) s between the indices of the two terms that are coupled with $a(n) = d_s a(n-s)$ we have for example:

2.1. Stride 1.

$$a(n) = d_1 a(n-1); \quad a(0) = c_0; \tag{5}$$

$$A(x) = \frac{c_0}{1 - d_1 x}. \tag{6}$$

2.2. Stride 2.

$$a(n) = d_2 a(n-2); \quad a(0) = c_0; \quad a(1) = c_1; \tag{7}$$

$$A(x) = \frac{c_0 + c_1 x}{1 - d_2 x^2}. \tag{8}$$

If d_2 is a positive square, a decomposition in partial fractions might be useful:

$$a(n) = k_2^2 a(n-2); \quad a(0) = c_0; \quad a(1) = c_1; \tag{9}$$

$$A(x) = \frac{c_0 + c_1 x}{(1-k_2 x)(1+k_2 x)} = \frac{c_0 k_2 - c_1}{2k_2} \frac{1}{1+k_2 x} + \frac{c_0 k_2 + c_1}{2k_2} \frac{1}{1-k_2 x}. \tag{10}$$

2.3. **Stride 3.**

$$a(n) = d_3 a(n-3); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \tag{11}$$

$$A(x) = \frac{c_0 + c_1 x + c_2 x^2}{1 - d_3 x^3}. \tag{12}$$

If $d_3 \equiv k_3^3$ is a cube, the follow-up decomposition in partial fractions is

$$a(n) = k_3^3 a(n-3); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \tag{13}$$

$$\begin{aligned} A(x) &= \frac{c_0 + c_1 x + c_2 x^2}{(1 - k_3 x)(1 + k_3 x + k_3^2 x^2)} \\ &= \frac{1}{3k_3^2} \frac{2c_0 k_3^2 - c_1 k_3 - c_2 + (c_0 k_3^3 + c_1 k_3^2 - 2c_2 k_3)x}{1 + k_3 x + k_3^2 x^2} + \frac{1}{3k_3^2} \frac{c_0 k_3^2 + c_1 k_3 + c_2}{1 - k_3 x}. \end{aligned} \tag{14}$$

2.4. **Stride 4.**

$$a(n) = d_4 a(n-4); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \quad a(3) = c_3; \tag{15}$$

$$A(x) = \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - d_4 x^4}. \tag{16}$$

The case of $d_4 = k_4^2$ being a square is in particular

$$a(n) = k_4^2 a(n-4); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \quad a(3) = c_3; \tag{17}$$

$$A(x) = \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{(1 - k_4 x^2)(1 + k_4 x^2)} = \frac{1}{2k_4} \frac{c_0 k_4 + c_2 + (c_1 k_4 + c_3)x}{1 - k_4 x^2} + \frac{1}{2k_4} \frac{c_0 k_4 - c_2 + (c_1 k_4 - c_3)x}{1 + k_4 x^2}. \tag{18}$$

2.5. **General.** The obvious pattern is

$$a(n) = d_s a(n-s); \quad a(i) = c_i; \quad 0 \le i < s; \tag{19}$$

$$A(x) = \frac{\sum_{i=0}^{s-1} c_i x^i}{1 - d_s x^s}. \tag{20}$$

3. 2-Term Homogeneous

This is the most busy case [11]:

$$a(n) = d_1 a(n-1) + d_2 a(n-2); \quad a(0) = c_0; \quad a(1) = c_1; \tag{21}$$

$$A(x) = \frac{c_0 + (c_1 - d_1 c_0)x}{1 - d_1 x - d_2 x^2}. \tag{22}$$

The case $d_1 = 0$ reduces to (8).

3.1. **First Term Not Coupled.**

$$a(n) = d_2 a(n-2) + d_3 a(n-3); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \tag{23}$$

$$A(x) = \frac{c_0 + c_1 x + (c_2 - c_0 d_2)x^2}{1 - d_2 x^2 - d_3 x^3}. \tag{24}$$

3.2. **Second Term Not Coupled.**

$$a(n) = d_1 a(n-1) + d_3 a(n-3); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \tag{25}$$

$$A(x) = \frac{c_0 + (c_1 - c_0 d_1)x + (c_2 - c_1 d_1)x^2}{1 - d_1 x - d_3 x^3}. \tag{26}$$

3.3. **First Two Terms Not Coupled.**

$$a(n) = d_3 a(n-3) + d_4 a(n-4); \quad a(0) = c_0; \quad a(1) = c_1; \quad a(2) = c_2; \quad a(3) = c_3; \tag{27}$$

$$A(x) = \frac{c_0 + c_1 x + c_2 x^2 + (c_3 - c_0 d_3)x^3}{1 - d_3 x^3 - d_4 x^4}. \tag{28}$$

3.4. **First $s-1$ Terms Not Coupled.**

(22), (24) and (28) are special cases of

$$a(n) = d_s a(n-s) + d_{s+1} a(n-s-1); \quad a(i) = c_i; \quad 0 \le i \le s; \quad s \ge 1; \tag{29}$$

$$A(x) = \frac{\sum_{i=0}^{s} c_i x^i - c_0 d_s x^s}{1 - d_s x^s - d_{s+1} x^{s+1}}. \tag{30}$$

3.5. **Bisections.**

If the denominator is a polynomial in a higher power of x, the sequence is an overlay of de-facto decoupled subsequences. Consider for example the generating function

$$A(x) = \frac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - d_2 x^2 - d_4 x^4} \tag{31}$$

which has no terms $\propto x$ or $\propto x^3$ in the denominator. This defines two subsequences at even and odd indices of the form

$$a(2n) = d_2 a(2n-2) + d_4 a(2n-4); \tag{32}$$

$$a(2n-1) = d_2 a(2n-3) + d_4 a(2n-5), \tag{33}$$

with initial values $a(0)$ and $a(2)$ for the even terms and $a(1)$ and $a(3)$ for the odd terms. We show how the 6 parameters [four initial values $a(0..3)$ and two coefficients d] can be reorganized as

$$a(2n) = \beta_{1e} a(2n-1) + \beta_{2e} a(2n-2); \tag{34}$$

$$a(2n-1) = \beta_{1o} a(2n-2) + \beta_{2o} a(2n-3), \tag{35}$$

with 6 parameters [four coefficients β and 2 initial values $a(0)$ and $a(1)$] that mix the two subsequences. The β are obtained as follows. Splitting $A(x)$ in the even function $(c_0 + c_2 x^2)/(1 - d_2 x^2 - d_4 x^4)$ and the odd function $(c_1 x + c_3 x^3)/(1 - d_2 x^2 - d_4 x^4)$ generates for even indices

$$\begin{aligned} a(2n) = [x^{2n}]A(x) &= c_0 [x^{2n}] \frac{1}{1 - d_2 x^2 - d_4 x^4} + c_2 [x^{2n}] \frac{x^2}{1 - d_2 x^2 - d_4 x^4} \\ &= c_0 [x^{2n}] \frac{1}{1 - d_2 x^2 - d_4 x^4} + c_2 [x^{2n-2}] \frac{1}{1 - d_2 x^2 - d_4 x^4}; \end{aligned} \tag{36}$$

$$a(2n-2) = c_0 [x^{2n-2}] \frac{1}{1 - d_2 x^2 - d_4 x^4} + c_2 [x^{2n-4}] \frac{1}{1 - d_2 x^2 - d_4 x^4}, \tag{37}$$

and for odd indices

$$a(2n-1) = [x^{2n-1}]A(x) = c_1[x^{2n-1}]\frac{x}{1-d_2x^2-d_4x^4} + c_3[x^{2n-1}]\frac{x^3}{1-d_2x^2-d_4x^4}$$

$$(38) \qquad = c_1[x^{2n-2}]\frac{1}{1-d_2x^2-d_4x^4} + c_3[x^{2n-4}]\frac{1}{1-d_2x^2-d_4x^4}.$$

The previous two equations are a linear 2×2 system of equations for $[x^{2n-2}]\frac{1}{1-d_2x^2-d_4x^4}$ and $[x^{2n-4}]\frac{1}{1-d_2x^2-d_4x^4}$ which is solved by

(39)
$$[x^{2n-2}]\frac{1}{1-d_2x^2-d_4x^4} = \begin{vmatrix} a(2n-2) & c_2 \\ a(2n-1) & c_3 \end{vmatrix} / \begin{vmatrix} c_0 & c_2 \\ c_1 & c_3 \end{vmatrix} = \frac{c_3a(2n-2)-c_2a(2n-1)}{c_3c_0-c_2c_1};$$

(40)
$$[x^{2n-4}]\frac{1}{1-d_2x^2-d_4x^4} = \begin{vmatrix} c_0 & a(2n-2) \\ c_1 & a(2n-1) \end{vmatrix} / \begin{vmatrix} c_0 & c_2 \\ c_1 & c_3 \end{vmatrix} = \frac{c_0a(2n-1)-c_1a(2n-2)}{c_3c_0-c_2c_1}.$$

We insert the generic recurrence for the auxiliary sequence $1, 0, d_2, 0, d_4, 0, d_2^2 + d_4, 0, d_2^3 + 2d_2d_4, \ldots$,

(41)
$$[x^{2n}]\frac{1}{1-d_2x^2-d_4x^4} = d_2[x^{2n-2}]\frac{1}{1-d_2x^2-d_4x^4} + d_4[x^{2n-4}]\frac{1}{1-d_2x^2-d_4x^4}$$

in the right hand side of (36)

$$(42) \quad a(2n) = (c_0d_2+c_2)[x^{2n-2}]\frac{1}{1-d_2x^2-d_4x^4} + c_0d_4[x^{2n-4}]\frac{1}{1-d_2x^2-d_4x^4},$$

and then (39) and (40)

$$(43) \quad = (c_0d_2+c_2)\frac{c_3a(2n-2)-c_2a(2n-1)}{c_3c_0-c_2c_1} + c_0d_4\frac{c_0a(2n-1)-c_1a(2n-2)}{c_3c_0-c_2c_1}.$$

By comparison with the form (34) we conclude

$$(44) \qquad \beta_{1e} = \frac{c_0^2d_4 - c_2^2 - c_0d_2c_2}{c_3c_0 - c_1c_2},$$

$$(45) \qquad \beta_{2e} = \frac{c_3c_0d_2 - c_1c_0d_4 + c_3c_2}{c_3c_0 - c_1c_2}.$$

The equivalent computation for the odd indices yields

$$(46) \qquad \beta_{1o} = \frac{c_1^2d_4 - c_1c_3d_2 - c_3^2}{c_1c_0d_4 - c_3c_2 - c_3c_0d_2},$$

$$(47) \qquad \beta_{2o} = \frac{d_4(c_3c_0 - c_1c_2)}{c_1c_0d_4 - c_3c_2 - c_3c_0d_2}.$$

4. Inhomogeneous

With (4), calculation of $A(x)$ reduces to the calculation of $B(x)$, that is, to looking at the simpler format

$$(48) \qquad a(n) = b(n).$$

4.1. **Simple Powers.** If $b(n)$ is a linear combination of nth powers with constant coefficients with optional offsets o_j,

$$a(n) = \sum_{j=0} d_j b_j^{n-o_j}, \tag{49}$$

where neither the d_j nor the b_j nor the o_j depend on n, the generating function is the associated geometric series [1, 3.1.10]

$$A(x) = \sum_{j=0} d_j b_j^{-o_j} \frac{1}{1-b_j x}. \tag{50}$$

4.2. **Polynomials.** The case of the constant term

$$a(n) = 1 \tag{51}$$

is the simplest form of (50) with the generating function [1, 3.6.10]

$$A(x) = \frac{1}{1-x}. \tag{52}$$

k-fold differentiation with respect to x computes the generating functions of kth order polynomials of n of the format

$$a(n) = n(n-1)(n-2)\cdots(n-k+1) = n!/(n-k)!, \tag{53}$$

$$A(x) = \frac{k!x^k}{(1-x)^{k+1}}; \quad k = 0, 1, 2, \ldots. \tag{54}$$

(See [17, (1.1)] for the determination of the exponential generating function along the same lines.) Decomposition of the general kth order polynomial into a sum of polynomials of this special kind by aid of the Stirling Numbers of the Second Kind $\mathcal{S}$ [1, 24.1.4] pairs the polynomial

$$b(n) = \sum_{j=0} e_j n^j \tag{55}$$

with constant coefficients e_j with the generating function

$$A(x) = \sum_{j=0} e_j \sum_{k=0}^{j} \mathcal{S}_j^{(k)} k! \frac{x^k}{(1-x)^{k+1}}. \tag{56}$$

The same methodology of repeated differentiation with respect to x may be applied to the more general (50) and allows construction of generating functions for $a(n) = \sum_{j=0} \sum_{k=0} e_{k,j} n^k b_j^n$, sums of products of simple powers and polynomials.

5. Transformation of Series

5.1. **Multisection, Delta-Operator.** Generating functions of

- multisections of sequences and the inverse—which is a shuffling operation of many sequences into one—[6]
- first and higher order differences

are implemented as described by Riordan [15].

5.2. **Binomial Transform.** The (inverse) binomial transform relates two sequences $a(n)$ and $b(n)$ via

$$a(n) \equiv \sum_{k=0}^{n} \binom{n}{k} b(k); \quad b(n) \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a(k);, \tag{57}$$

which induces a relation between the generating functions $A(x) \equiv \sum_n a(n)x^n$ and $B(x) \equiv \sum_n b(n)x^n$ as follows [3, 7, 8, 21, 20]:

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b(k) x^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} b(k) x^n \\ &= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \binom{s+k}{k} b(k) x^{s+k} = \sum_{k=0}^{\infty} b(k) x^k \sum_{s=0}^{\infty} \binom{s+k}{k} x^s \\ &= \sum_{k=0}^{\infty} b(k) x^k \frac{1}{(1-x)^{k+1}} = \frac{1}{1-x} \sum_{k=0}^{\infty} b(k) \left(\frac{x}{1-x}\right)^k = \frac{1}{1-x} B\left(\frac{x}{1-x}\right). \end{aligned} \tag{58}$$

$$B(x) = \frac{1}{1+x} A\left(\frac{x}{1+x}\right). \tag{59}$$

These methods can be chained to provide generating functions of other transforms [18, 19, 22].

References

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642 (29 #4914)
2. Roberto Bagnara, *Parma university's recurrence relation solver*, arXiv:cs/0512056.
3. Mira Bernstein and Neil J. A. Sloane, *Some canonical sequences of integers*, Lin. Alg. Applic. **226–228** (1995), 57–72, (E:) [4]. MR 1344554 (96i:05004)
4. Richard A. Brualdi, *From the editor-in-chief*, Lin. Alg. Applic. **320** (2000), no. 1–3, 209–216. MR 1796542
5. Huantian Cao, *Autogf: an automated system to calculate coefficients of generating functions*, Master's thesis, Massachusetts Institute of Technology, 2002.
6. Wenchang Chu, *Some binomial convolution formulas*, Fib. Quart. **40** (2002), no. 1, 19–32.
7. Ayhan Dil and István Mezö, *A symmetric algorithm for hyperharmonic and fibonacci numbers*, arXiv:0803.4388 [math.NT] (2008).
8. ———, *A symmetric algorithm for hyperharmonic and Fibonacci numbers*, Appl. Math. Comput. **206** (2008), no. 2, 942–951. MR 2483070
9. Ashok Kumar Gupta and Ashok Kumar Mittal, *Integer sequences associated with integer monic polynomial*, arXiv:math.GM/0001112 (2000).
10. Subhashk Kak, *The golden mean and the physics of aesthetics*, arXiv:physics/0411195 (2004).
11. István Mező, *Several generating functions for second-order recurrence sequences*, J. Integer Seq. (2009), no. 12, 09.3.8. MR 2500953 (2010c:11019)
12. Tang Minh and Tan Van To, *Using generating functions to solve linear inhomogeneous recurrence equations*, Int. Conf. Simulation, Modelling and Optimization, vol. 6, 2006, p. 399.
13. G. Myerson and A. J. van der Poorten, *Some problems concerning recurrence sequences*, Amer. Math. Monthly **102** (1995), no. 8, 698–705. MR 1357486 (97a:11029)
14. Simon Plouffe, *1031 generating functions and conjectures*, 1992.
15. John Riordan, *Combinatorial identities*, John Wiley, New York, 1968. MR 0231725 (38 #53)
16. Bruno Salvy and Paul Zimmerman, *Gfun: a maple package for the manipulation of generating and holonomic functions in one variable*, ACM Trans. Math. Softw. **20** (1994), no. 2, 163–177.
17. Susumu Shirai and Ken ichi Sato, *Some identities involving Bernoulli and Stirling numbers*, J. Number Theory **90** (2001), no. 1, 130–142.
18. Michael Z. Spivey, *Combinatorial sums and finite differences*, Discrete Math. **307** (2007), no. 24, 3130–3146. MR 2370116 (2008j:05013)

19. Michael Stoll, *Bounds for the length of recurrence relations for convolutions of p-recursive sequences*, Eur. J. Comb. **18** (1997), no. 6, 707. MR 1468339 (99f:05007)
20. Zhi-Hong Sun, *Invariant sequences under binomial transformation*, Fib. Quart. **39** (2001), no. 4, 324–333. MR 1851531 (2002f:11012)
21. Stefan Weinzierl, *Expansion around half-integer values, binomial sums, and inverse binomial sums*, J. Math. Phys **45** (2004), no. 7, 2656–2673. MR 2067580 (2005f:33042)
22. P. Wynn, *A note on the generalised Euler transformation*, Comp. J. **14** (1971), no. 4, 437–441. MR 0321266 (47 #9799)

URL: http://www.mpia.de/~mathar/public/mathar20071126.pdf

Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany