Astronomical Air Mass

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The standard formula for the air mass (in Astronomy) is one divided by the cosine of the zenith angle. We compute corrections to this formula assuming a finite earth radius and an exponential scale height.

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I. SCOPE

Astronomers define the air mass not by the 1 gram per liter (at standard temperature and pressure) but as a measure of how much air of the atmosphere is in the line of sight between the telescope and the kind of vacuum higher then, say, 20 km of altitude.

Let $\rho(h)$ be the density of air at altitude h above the telescope; the areal density is given by integration along the direction of the light rays

$$A_{[\rho]} = \int \rho(h) dx. \tag{1}$$

(Note that we have not specified whether ρ is a mass density or number density or some refractive index.) The value of A has a major influence on

1. the amount of absorption of the star light by the atmosphere;
2. the change of the angle of arrival by refraction and its dispersion effects (also known as the "transverse" atmospheric dispersion);
3. the amount of turbulence observed in the pictures by the rapid density fluctuations in the air.

II. FLAT GEOMETRY

If z denotes the zenith angle of the pointing (the 90° complement of the star's altitude above the horizon), a substitution of the integration variable in (1) by $x = h/\cos z$, $dx = dh/\cos z$ as in Figure 1 yields the air mass

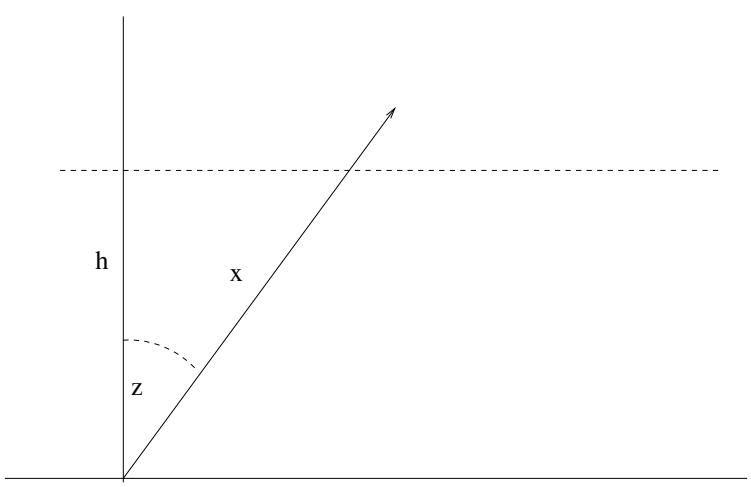

FIG. 1. The relation $\cos z = h/x$.

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$$a \equiv \frac{A_{[\rho]}(z)}{A_{[\rho]}(0)} = \frac{\int \rho(h) \frac{dh}{\cos z}}{\int \rho(h) \frac{dh}{\cos 0}} = \frac{1}{\cos z}. \tag{2}$$

This is the standard expression. Fortunately this result is independent of the structure of the density function $\rho(h)$. In the limit of observing near the horizon, $z \to 90°$, the value becomes infinite, indicating that the view through the telescope never leaves the atmosphere.

III. FINITE EARTH CURVATURE

A. Taylor Expansion for Small Relative Atmospheric Height

An improved calculation recognizes an effective Earth Radius R—the Earth radius plus the altitude of the observatory above sea level—. Consider the triangle of the edges defined by (i) the observer, (ii) earth center, and a (iii) point on the straight line of sight at distance x as in Figure 2.

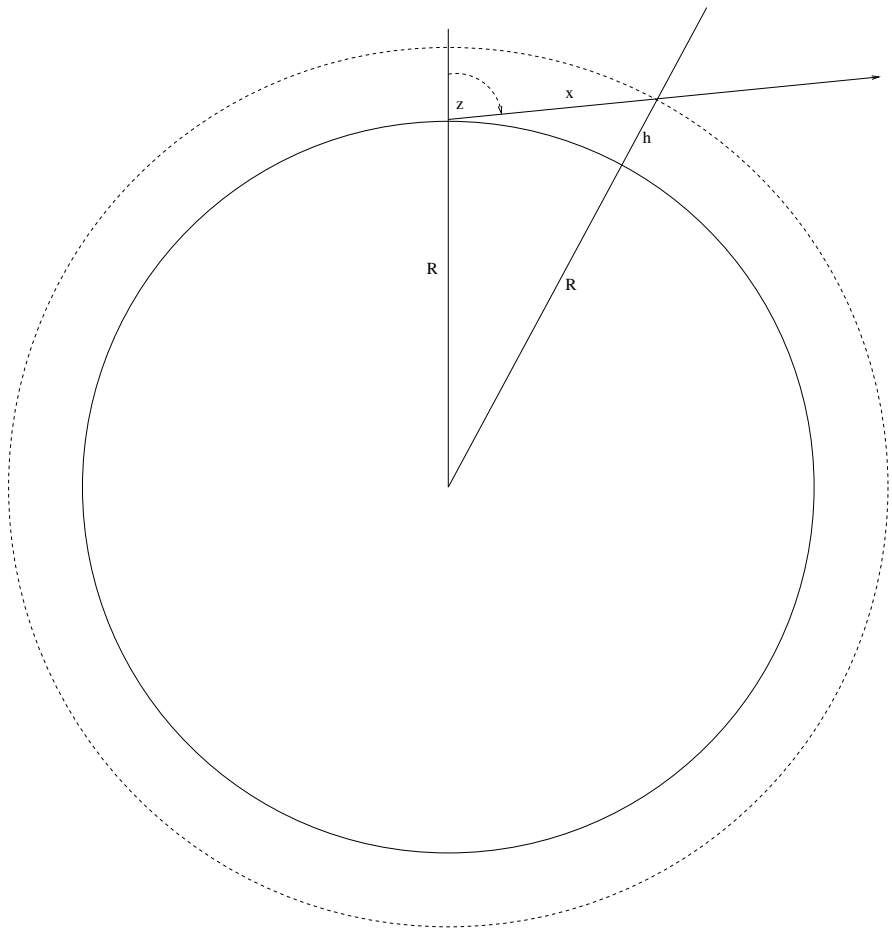

FIG. 2. The triangle with edges of length R, x and $R + h$ in Equation (3).

The interior angle at the observer's position is the complementary $\pi - z$, and the law of cosines is [1, 4.3.148]

$$\cos(\pi - z) = \frac{x^2 + R^2 - (R + h)^2}{2xR}; \tag{3}$$

$$-2xR\cos(z) = x^2 - 2Rh - h^2. \tag{4}$$

Solving this quadratic equation for x gives

$$\frac{x}{R} = -\cos z + \cos z \sqrt{1 + 2\frac{h}{R}\frac{1}{\cos^2 z} + (\frac{h}{R})^2 \frac{1}{\cos^2 z}}. \tag{5}$$

The atmospheric thickness of the order of 10 km in relation to the Earth Radius of 6300 km implies we may introduce a Taylor expansion in terms of the small h/R:

$$x \approx \frac{R}{\cos z}(h/R) + \frac{R}{\cos z}\frac{\cos^2 z - 1}{2\cos^2 z}(h/R)^2 - \frac{R}{\cos z}\frac{\cos^2 z - 1}{2\cos^4 z}(h/R)^3 - \frac{R}{\cos z}\frac{(\cos^2 z - 1)(\cos^2 z - 5)}{8\cos^6 z}(h/R)^4 + \cdots$$

$$= \frac{h}{\cos z} - \frac{R}{\cos z}\frac{\tan^2 z}{2}(h/R)^2 + \frac{R}{\cos z}\frac{\tan^2 z}{2\cos^2 z}(h/R)^3 - \frac{R}{\cos z}\frac{\tan^2 z(4+\sin^2 z)}{8\cos^4 z}(h/R)^4 + \cdots \tag{6}$$

(In essence this is also an expansion in terms of odd powers of $1/\cos z$.) The substitution of the differential in (1) is

$$\frac{dx}{dh} = \frac{R+h}{\sqrt{R^2\cos^2 z + 2hR + h^2}}, \tag{7}$$

and the approximate substitution of the differential is

$$\frac{dx}{dh} \approx \frac{1}{\cos z} - \frac{1}{\cos z}\frac{\tan^2 z}{2}2(h/R) + \frac{1}{\cos z}\frac{\tan^2 z}{2\cos^2 z}3(h/R)^2 - \frac{1}{\cos z}\frac{\tan^2 z(4+\sin^2 z)}{8\cos^4 z}4(h/R)^3 + \cdots$$

$$= \frac{1}{\cos z} - \frac{1}{\cos z}\tan^2 z(h/R) + \frac{1}{\cos z}\frac{3\tan^2 z}{2\cos^2 z}(h/R)^2 - \frac{1}{\cos z}\frac{\tan^2 z(4+\sin^2 z)}{2\cos^4 z}(h/R)^3 + \cdots \tag{8}$$

The integral (1) becomes

$$A \approx \frac{1}{\cos z}\left[\int_0^\infty \rho(h)dh - \tan^2 z\int_0^\infty \rho(h)(h/R)dh + \frac{3\tan^2 z}{2\cos^2 z}\int_0^\infty \rho(h)(h/R)^2 dh - \frac{\tan^2 z(4+\sin^2 z)}{2\cos^4 z}\int_0^\infty \rho(h)(h/R)^3 dh + \cdots\right] \tag{9}$$

A standard model is an exponential decay of the air density with an atmospheric height of $K \approx 9.6$ km [2]:

$$\rho = \rho_0 e^{-h/K}. \tag{10}$$

The j-th order correction in comparison to the expression of Section II is [3, 3.351.3]

$$\int_0^\infty \rho(h)(h/R)^j dh = \frac{\rho_0}{R^j}\int_0^\infty e^{-h/K}h^j dh = j!\frac{\rho_0}{R^j}K^{j+1}. \tag{11}$$

Inserted into A this yields

$$A \approx \frac{\rho_0}{\cos z}[K - \tan^2 z\frac{K^2}{R} + \frac{3\tan^2 z}{2\cos^2 z}\frac{2K^3}{R^2} - \frac{\tan^2 z(4+\sin^2 z)}{2\cos^4 z}\frac{6K^4}{R^3}]$$

$$= \frac{\rho_0 K}{\cos z}[1 - \tan^2 z(\frac{K}{R}) + \frac{3\tan^2 z}{\cos^2 z}(\frac{K}{R})^2 - 3\frac{\tan^2 z(4+\sin^2 z)}{\cos^4 z}(\frac{K}{R})^3] \tag{12}$$

where $K/R \approx 1.5 \times 10^{-3}$. The air mass becomes

$$a \approx \frac{1}{\cos z}[1 - \tan^2 z(\frac{K}{R}) + \frac{3\tan^2 z}{\cos^2 z}(\frac{K}{R})^2 - 3\frac{\tan^2 z(4+\sin^2 z)}{\cos^4 z}(\frac{K}{R})^3]. \tag{13}$$

B. At the Horizon

The taylor expansion of (7) for small $\cos z$ reads

$$\frac{dx}{dh} = \frac{1+h/R}{\sqrt{2(h/R)+(h/R)^2}} - \frac{1+h/R}{2[2(h/R)+(h/R)^2]^{3/2}}\cos^2 z + \frac{3}{8}\frac{1+h/R}{(2[h/R)+(h/R)^2]^{5/2}}\cos^4 z + O(\cos^6 z). \tag{14}$$

And in the exponential model the integral along the line of sight is also expanded in powers of h/R. For $\cos z = 0$ this becomes

$$\begin{aligned} A &= \rho_0 \int_0^\infty e^{-h/K} \frac{1}{\sqrt{h/R}} \frac{1+h/R}{\sqrt{2+h/R}} dh \\ &\approx \rho_0 \int_0^\infty e^{-h/K} \frac{1}{\sqrt{h/R}} \sqrt{2} \left[\frac{1}{2} + \frac{3}{8}(h/R) - \frac{5}{64}(h/R)^2 + \frac{7}{256}(h/R)^3 + \cdots \right] dh \\ &= \sqrt{2}\rho_0 \left[\frac{1}{2}\sqrt{\pi R K} + \frac{3K}{16}\sqrt{\frac{K\pi}{R}} - \frac{15K^2}{256R}\sqrt{\frac{K\pi}{R}} + \cdots \right]. \end{aligned} \tag{15}$$

Dividing through $A(z=0) = \rho_0 K$ gives for the air mass at $z = 90°$

$$a = \frac{1}{2}\sqrt{\frac{2\pi}{K/R}} + \frac{3}{16}\sqrt{\frac{2K\pi}{R}} - \frac{15K}{256R}\sqrt{\frac{2K\pi}{R}} + \cdots . \tag{16}$$

The problem with this approach is that the next term, $O(\cos^2 z)$ in (14), leads to a diverging integral if integrated over h, because it is $\sim h^{-3/2}$ for small h, and not just $\sim h^{-1/2}$ as the term of $O(\cos^0 z)$.

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