Astronomical Air Mass

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The standard formula for the air mass (in Astronomy) is one divided by the cosine of the zenith angle. We compute corrections to this formula assuming a finite earth radius and an exponential scale height.

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I. SCOPE

Astronomers define the air mass not by the 1 gram per liter (at standard temperature and pressure) but as a measure of how much air of the atmosphere is in the line of sight between the telescope and the kind of vacuum higher then, say, 20 km of altitude.

Let $\rho(h)$ be the density of air at altitude $h$ above the telescope; the areal density is given by integration along the direction of the light rays

$$A_{\rho} = \int \rho(h) dx.$$  \hfill (1)

(Note that we have not specified whether $\rho$ is a mass density or number density or some refractive index.) The value of $A$ has a major influence on

1. the amount of absorption of the star light by the atmosphere;
2. the change of the angle of arrival by refraction and its dispersion effects (also known as the “transverse” atmospheric dispersion);
3. the amount of turbulence observed in the pictures by the rapid density fluctuations in the air.

II. FLAT GEOMETRY

If $z$ denotes the zenith angle of the pointing (the 90° complement of the star’s altitude above the horizon), a substitution of the integration variable in (1) by $x = h/cos z$, $dx = dh/cos z$ as in Figure 1 yields the air mass

FIG. 1. The relation $cos z = h/x$.

http://www.mpia.de/~mathar
\[ a = \frac{A_{[\rho]}(z)}{A_{[\rho]}(0)} = \frac{\int \rho(h) \frac{dh}{\cos z}}{\int \rho(h) \frac{dh}{\cos 0}} = \frac{1}{\cos z}. \quad (2) \]

This is the standard expression. Fortunately this result is independent of the structure of the density function \( \rho(h) \). In the limit of observing near the horizon, \( z \to 90^\circ \), the value becomes infinite, indicating that the view through the telescope never leaves the atmosphere.

## III. FINITE EARTH CURVATURE

### A. Taylor Expansion for Small Relative Atmospheric Height

An improved calculation recognizes an effective Earth Radius \( R \)—the Earth radius plus the altitude of the observatory above sea level—. Consider the triangle of the edges defined by (i) the observer, (ii) earth center, and a (iii) point on the straight line of sight at distance \( x \) as in Figure 2.

![Figure 2](https://example.com/figure2.png)

**FIG. 2.** The triangle with edges of length \( R \), \( x \) and \( R + h \) in Equation (3).

The interior angle at the observer’s position is the complementary \( \pi - z \), and the law of cosines is [1, 4.3.148]

\[
\cos(\pi - z) = \frac{x^2 + R^2 - (R + h)^2}{2xR}; \quad (3)
\]

\[-2xR \cos(z) = x^2 - 2Rh - h^2. \quad (4)\]

Solving this quadratic equation for \( x \) gives

\[
\frac{x}{R} = -\cos z + \cos z \sqrt{1 + 2 \frac{h}{R} \frac{1}{\cos^2 z} + \left( \frac{h}{R} \right)^2 \frac{1}{\cos^2 z}}. \quad (5)
\]
The atmospheric thickness of the order of 10 km in relation to the Earth Radius of 6300 km implies we may introduce a Taylor expansion in terms of the small $h/R$:

\[ x \approx \frac{R}{\cos z} \left( \frac{h}{R} \right) + \frac{R}{\cos z} \frac{\cos^2 z - 1}{2 \cos^2 z} \left( \frac{h}{R} \right)^2 - \frac{R}{\cos z} \frac{\cos^2 z - 1}{2 \cos^4 z} \left( \frac{h}{R} \right)^3 - \frac{R}{\cos z} \frac{(\cos^2 z - 1)(\cos^2 z - 5)}{8 \cos^6 z} \left( \frac{h}{R} \right)^4 + \cdots \]

\[ = \frac{h}{\cos z} - \frac{R}{\cos z} \frac{\tan^2 z}{2} \left( \frac{h}{R} \right)^2 + \frac{R}{\cos z} \frac{\tan^2 z}{2 \cos^2 z} \left( \frac{h}{R} \right)^3 - \frac{R}{\cos z} \frac{\tan^2 z (4 + \sin^2 z)}{8 \cos^4 z} \left( \frac{h}{R} \right)^4 + \cdots \]  

(6)

(In essence this is also an expansion in terms of odd powers of $1/\cos z$.) The substitution of the differential in (1) is

\[ \frac{dx}{dh} \approx \frac{1}{\cos z} \frac{dx}{dh} = \frac{R + h}{\sqrt{R^2 \cos^2 z + 2hR + h^2}}, \]  

(7)

and the approximate substitution of the differential is

\[ \frac{dx}{dh} \approx \frac{1}{\cos z} \left( \frac{1}{\cos z} \frac{\tan^2 z}{2} 2 \left( \frac{h}{R} \right)^2 + \frac{1}{\cos z} \frac{\tan^2 z}{2 \cos^2 z} 3 \left( \frac{h}{R} \right)^2 - \frac{1}{\cos z} \frac{\tan^2 z (4 + \sin^2 z)}{8 \cos^4 z} \left( \frac{h}{R} \right)^3 + \cdots \right) \]

\[ = \frac{1}{\cos z} \frac{\tan^2 z \left( \frac{h}{R} \right) + 1}{\cos z} \frac{\tan^2 z (4 + \sin^2 z)}{2 \cos^4 z} \left( \frac{h}{R} \right)^3 + \cdots \]  

(8)

The integral (1) becomes

\[ A \approx \frac{1}{\cos z} \left[ \int_0^\infty \rho(h) dh - \tan^2 z \int_0^\infty \rho(h) \frac{h}{R} dh + \frac{3 \tan^2 z}{2 \cos^2 z} \int_0^\infty \rho(h) \frac{h}{R}^2 dh - \frac{\tan^2 z (4 + \sin^2 z)}{2 \cos^4 z} \int_0^\infty \rho(h) \frac{h}{R}^3 dh + \cdots \right] \]

A standard model is an exponential decay of the air density with an atmospheric height of $K \approx 9.6$ km [2]:

\[ \rho = \rho_0 e^{-h/K}. \]  

(10)

The $j$-th order correction in comparison to the expression of Section II is [3, 3.351.3]

\[ \int_0^\infty \rho(h) \frac{h}{R}^j dh = \frac{\rho_0}{R^j} \int_0^\infty e^{-h/K} h^j dh = j! \frac{\rho_0}{R^j} K^{j+1}. \]  

(11)

Inserted into $A$ this yields

\[ A \approx \frac{\rho_0}{\cos z} \left[ K - \tan^2 z \frac{K^2}{R^2} + \frac{3 \tan^2 z 2K^3}{2 \cos^2 z} \frac{K^2}{R^2} - \frac{\tan^2 z (4 + \sin^2 z) 6K^4}{2 \cos^4 z} \frac{K^2}{R^3} \right] \]

\[ = \frac{\rho_0 K}{\cos z} \left[ 1 - \tan^2 z \frac{K^2}{R^2} + \frac{3 \tan^2 z 2K^3}{\cos^2 z} \frac{K^2}{R^2} - \frac{3 \tan^2 z (4 + \sin^2 z) (K^3)}{\cos^4 z} \frac{K^2}{R^3} \right] \]  

(12)

where $K/R \approx 1.5 \times 10^{-3}$. The air mass becomes

\[ a \approx \frac{1}{\cos z} \left[ 1 - \tan^2 z \frac{K^2}{R^2} + \frac{3 \tan^2 z 2K^3}{\cos^2 z} \frac{K^2}{R^2} - \frac{3 \tan^2 z (4 + \sin^2 z) (K^3)}{\cos^4 z} \frac{K^2}{R^3} \right]. \]  

(13)

### B. At the Horizon

The taylor expansion of (7) for small $\cos z$ reads

\[ \frac{dx}{dh} = \frac{1 + h/R}{\sqrt{2(h/R) + (h/R)^2}} - \frac{1 + h/R}{2(2(h/R) + (h/R)^2)^{3/2}} \frac{\cos^2 z}{8 (2(h/R) + (h/R)^2)^{5/2}} + O(\cos^6 z). \]  

(14)
And in the exponential model the integral along the line of sight is also expanded in powers of $h/R$. For $\cos z = 0$ this becomes

$$A = \rho_0 \int_0^\infty e^{-h/K} \frac{1}{\sqrt{h/R}} \frac{1 + h/R}{\sqrt{1 + h/R}} dh$$

$$\approx \rho_0 \int_0^\infty e^{-h/K} \frac{1}{\sqrt{h/R}} \frac{1}{\sqrt{2}} \left[ \frac{1}{2} + \frac{3}{8} (h/R) - \frac{5}{64} (h/R)^2 + \frac{7}{256} (h/R)^3 + \cdots \right] dh$$

$$= \sqrt{2} \rho_0 \left[ \frac{1}{2} \sqrt{2 \pi RR} + \frac{3K}{16} \sqrt{\frac{K\pi}{R}} - \frac{15K^2}{256} \sqrt{\frac{K\pi}{R}} + \cdots \right]. \tag{15}$$

Dividing through $A(z = 0) = \rho_0 K$ gives for the air mass at $z = 90^\circ$

$$a = \frac{1}{2} \sqrt{\frac{2\pi}{K/R}} + \frac{3}{16} \sqrt{\frac{2K\pi}{R}} - \frac{15K}{256} \sqrt{\frac{2K\pi}{R}} + \cdots. \tag{16}$$

The problem with this approach is that the next term, $O(\cos^2 z)$ in (14), leads to a diverging integral if integrated over $h$, because it is $\sim h^{-3/2}$ for small $h$, and not just $\sim h^{-1/2}$ as the term of $O(\cos^0 z)$.