Animation of Karhunen-Loève Modes of Phase Screens

KL Mode mixing within the Taylor “frozen screen” approximation

Richard J. Mathar[1]

[1]Leiden Observatory (NEVEC)
University of Leiden

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Atmospheric Turbulence.

Turbulence: density, refractive indices.

Optics: “sharp” images: aim at maintaining a constant phase across M1 after passing through the atmosphere.

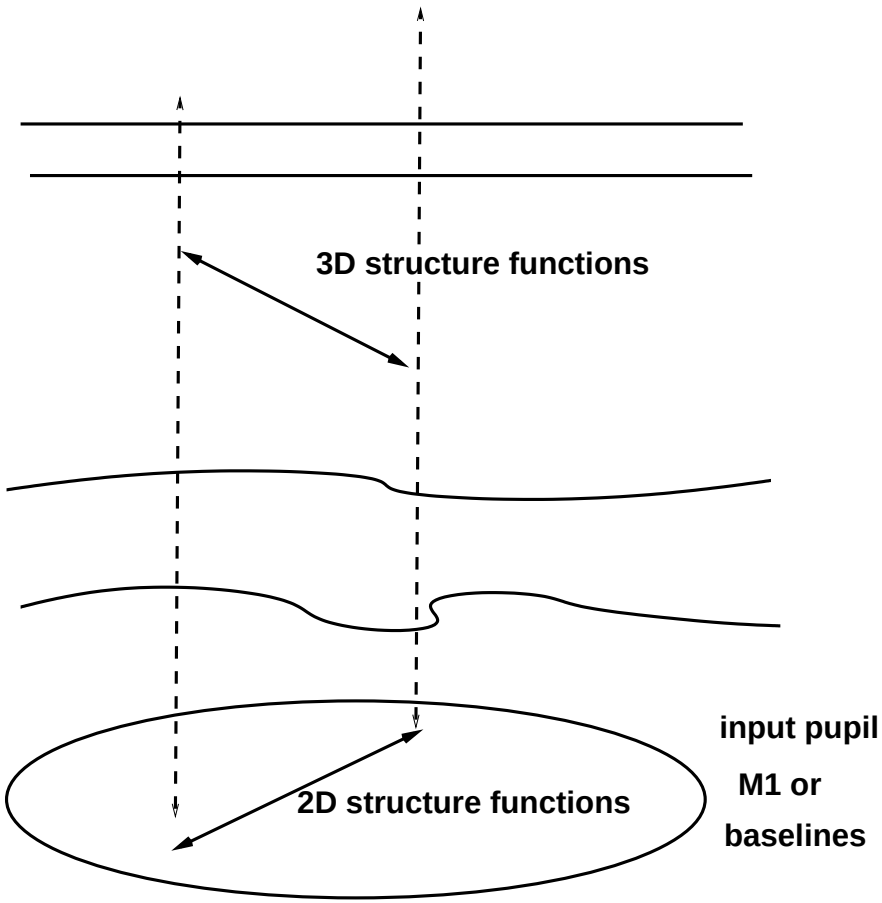
The Real Atmosphere is active optics: molecular densities fluctuate along the actual path (high Reynolds numbers)

- local thermal non-equilibrium $T(\mathbf{r})$
- inhomogeneous mixing (water vapor)

$\rightsquigarrow$ refractive indices $n(\lambda)$ fluctuate in time and space.

Atmospheric Turbulence.

Kolmogorov Power Law.



Structure Functions of molecular number density:

$$\mathcal{D}_\rho(\Delta\vec{x}) \equiv \langle \left|\rho(\vec{x}+\Delta\vec{x}) - \rho(\vec{x})\right|^2 \rangle$$

- 3D

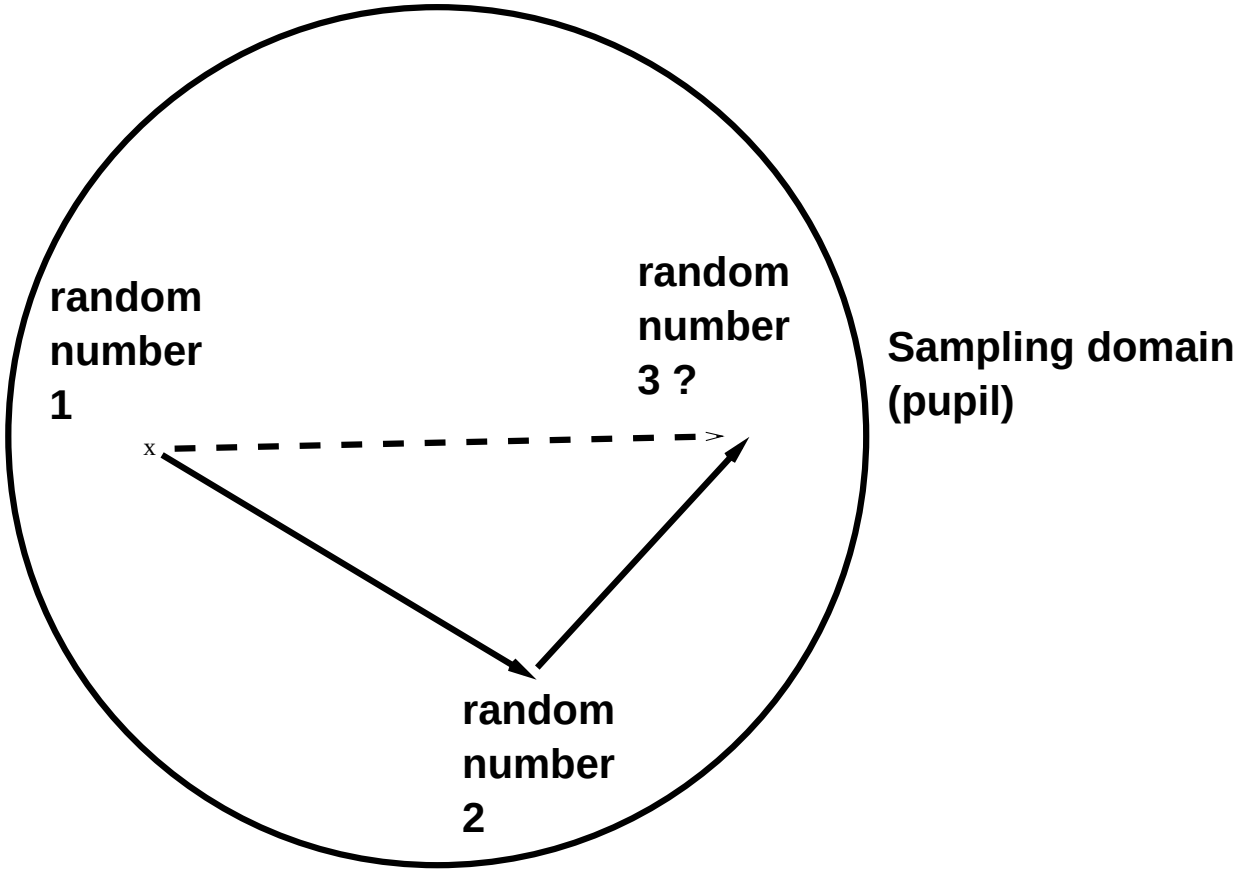
$$\mathcal{D}_\rho(\Delta\vec{x}) \propto (\Delta x)^{2/3}$$

- 2D, after integration through atmospheric layers, in pupil plane:

$$\mathcal{D}_\rho(\Delta\vec{x}) \propto (\Delta x)^{5/3}$$

Snapshot Simulation.

Sequential Scans over Pupil.



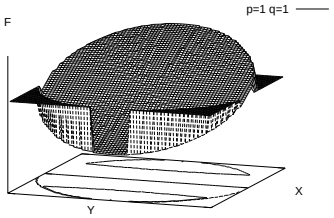
"Frustration" (in spin lattice speak) when dice are thrown in sequential manner to produce a phase screen spanning the sampling domain.

Snapshot Simulation.
KL modes.

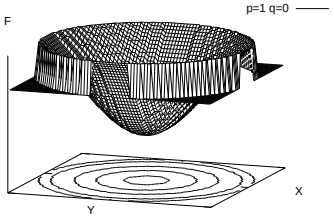
Solution: “Factorization” of the (i) expectation value / statistics and (ii) covariance ($\sim$ structure function) integral

Each phase screen instance is a stack (linear combination) of modes K:

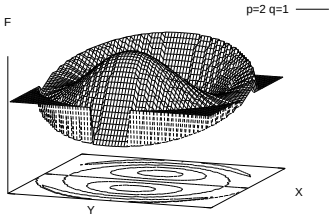
rand(1) $\times \lambda_1 \times$



+rand(2) $\times \lambda_2 \times$

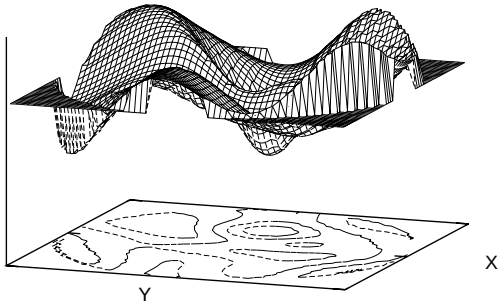


+rand(3) $\times \lambda_3 \times$



$+\ldots \quad \times \quad \times \qquad\qquad \ldots$

$\Big\} =$



Snapshot Simulation.
KL Eigenvalue Equation.

Modes K are Eigenfunctions of an Integral Operator

$$\int_{\text{domain,pupil}} d\vec{x}' \quad \underbrace{C(\vec{x},\vec{x}')}_{2-\text{point correlation}} \quad K_{p,q}(\vec{x}') = \underbrace{\lambda^2_{p,q}}_{\text{eigenvalue}} \quad K_{p,q}(\vec{x}).$$

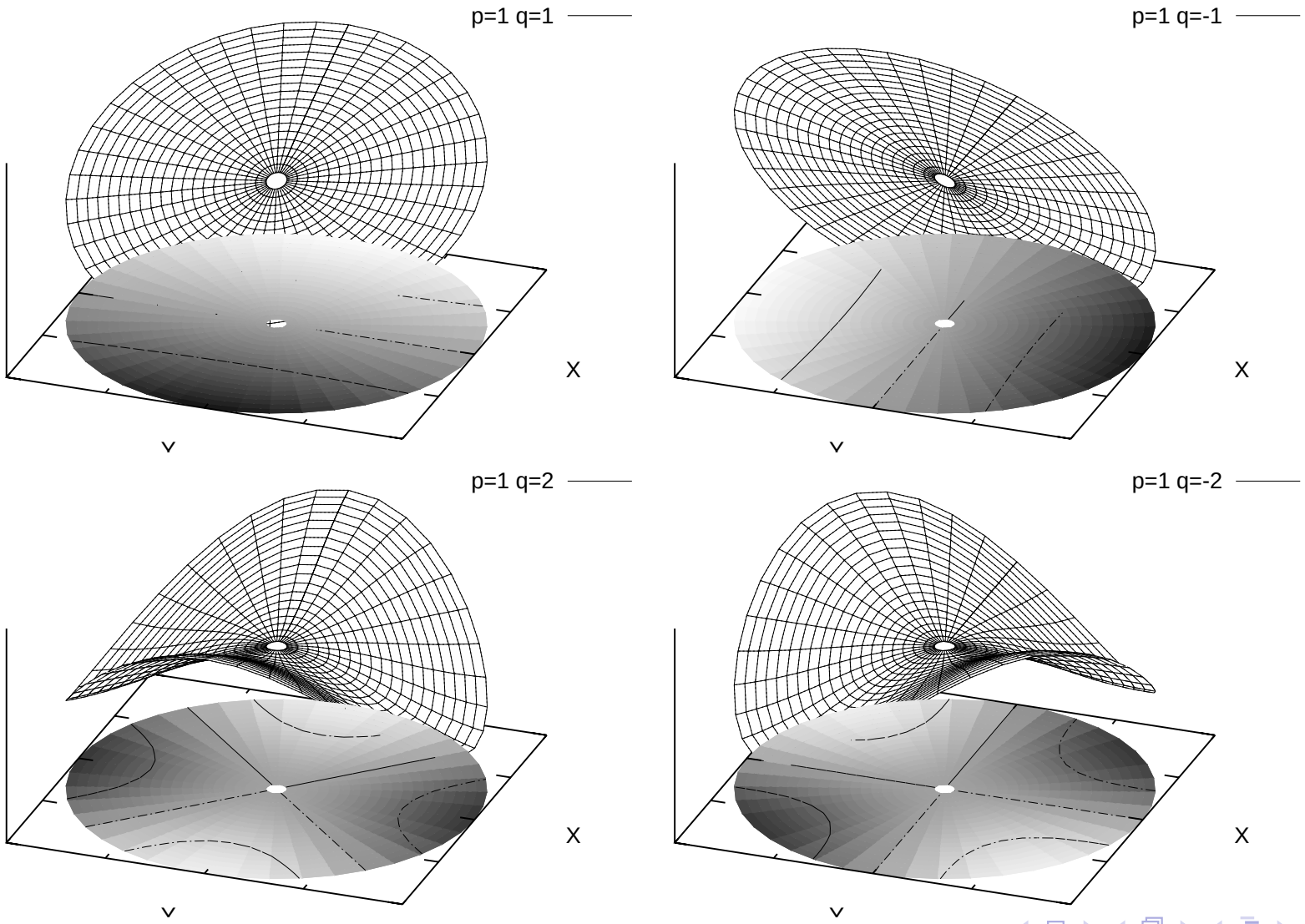
If correlation (isotropy) matches domain shape (circular): separation ansatz in azimuth and radial coordinates in the pupil [Fried 1978]:

$$K(\vec{x}) = \left.\begin{matrix}\sin(q\theta)\\ \cos(q\theta)\end{matrix}\right\} \times K(r), \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le \text{radius}$$

Efficient to work out the integral in wavenumber (Fourier) space for some basis functions (Zernike basis) [Roddier 1990].

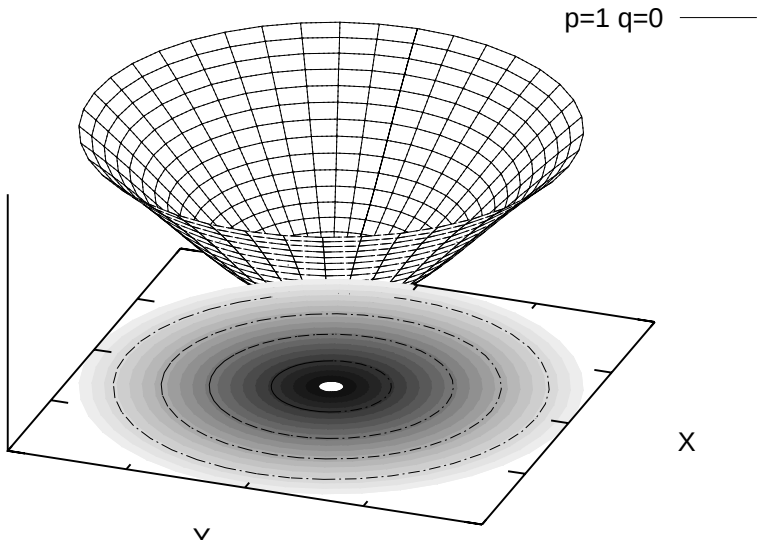
Snapshot Simulation.
KL Modes 1-4.

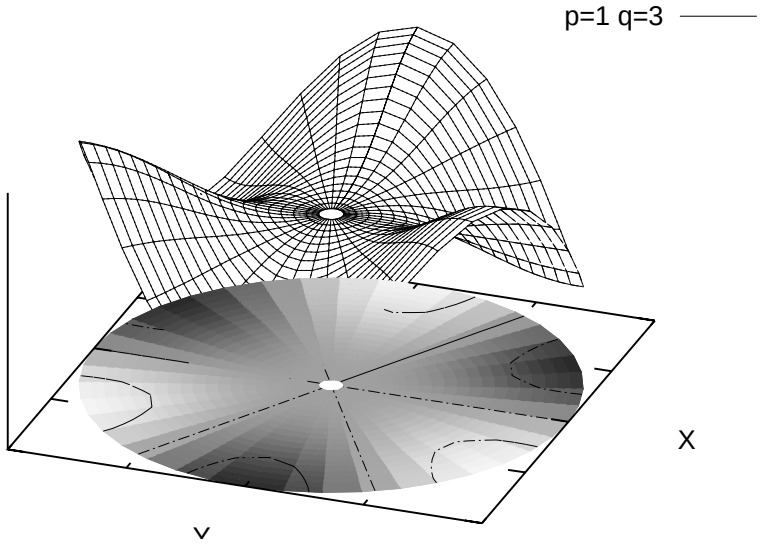
Approximately first 60 modes for MACAO/VLT simulation.

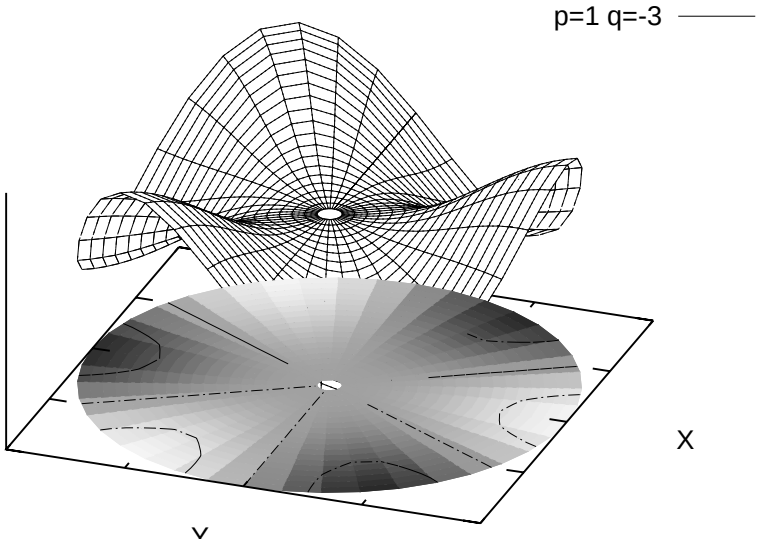


Snapshot Simulation.

KL Modes 5-7.

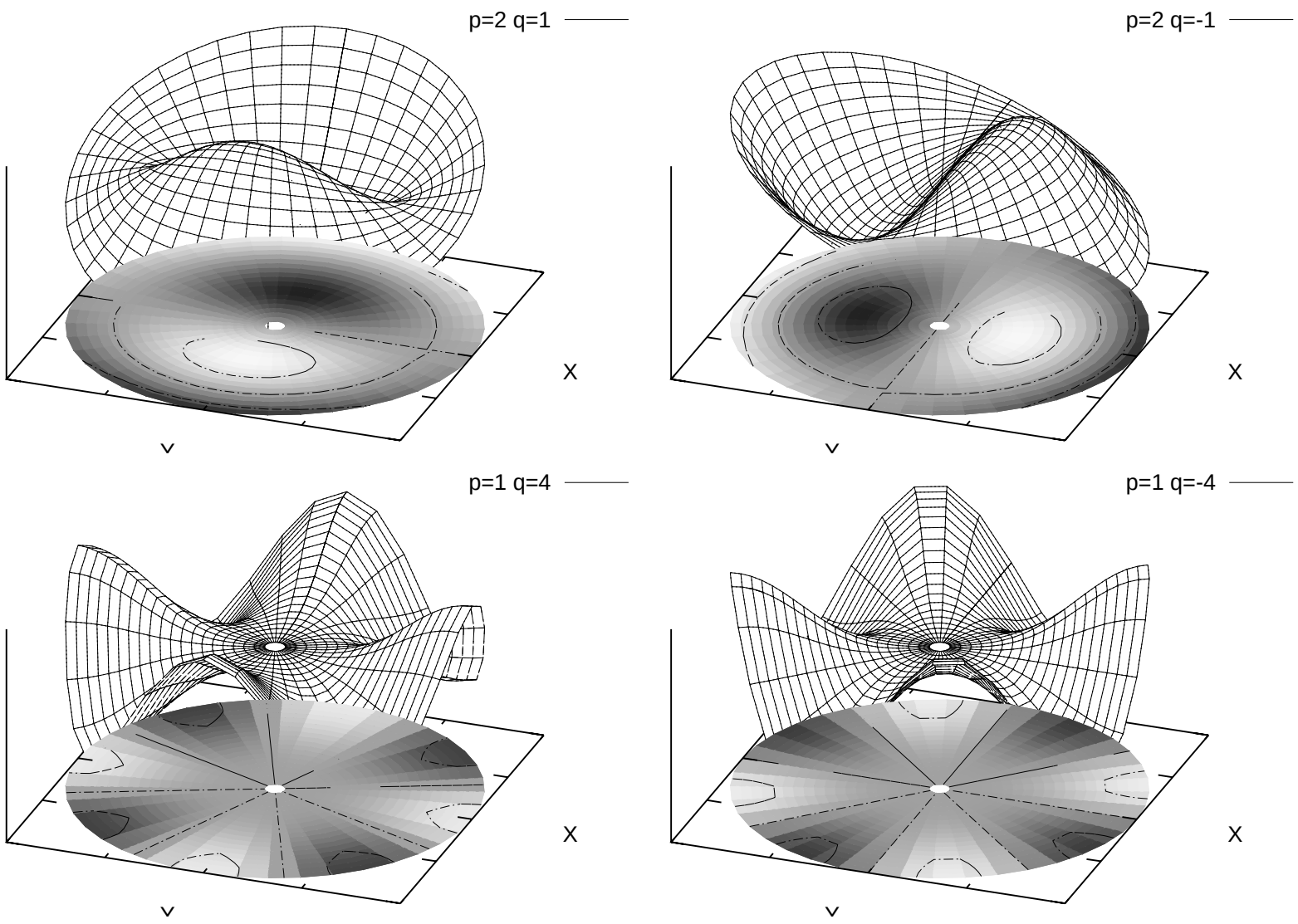






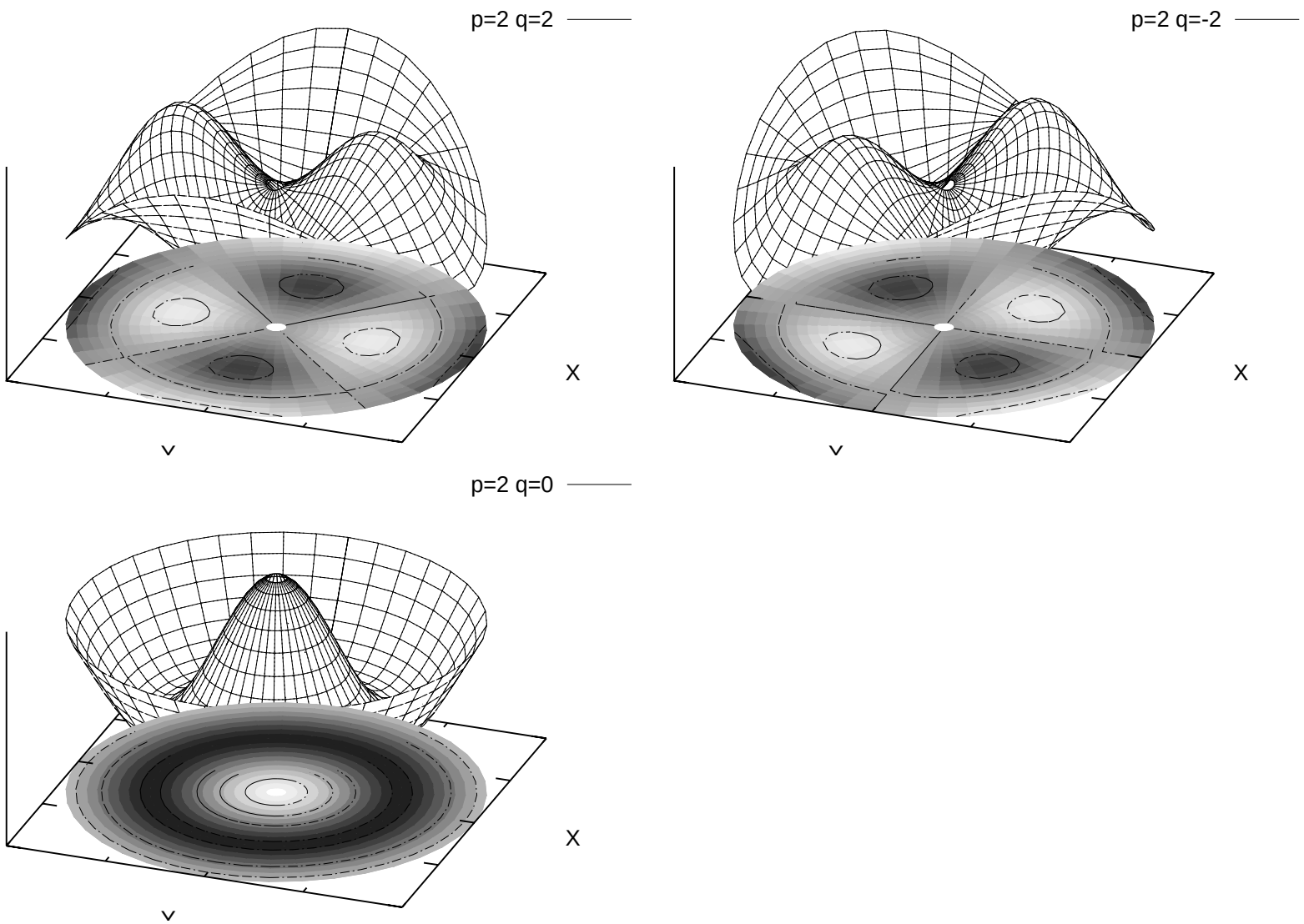
Snapshot Simulation.

KL Modes 8-11.



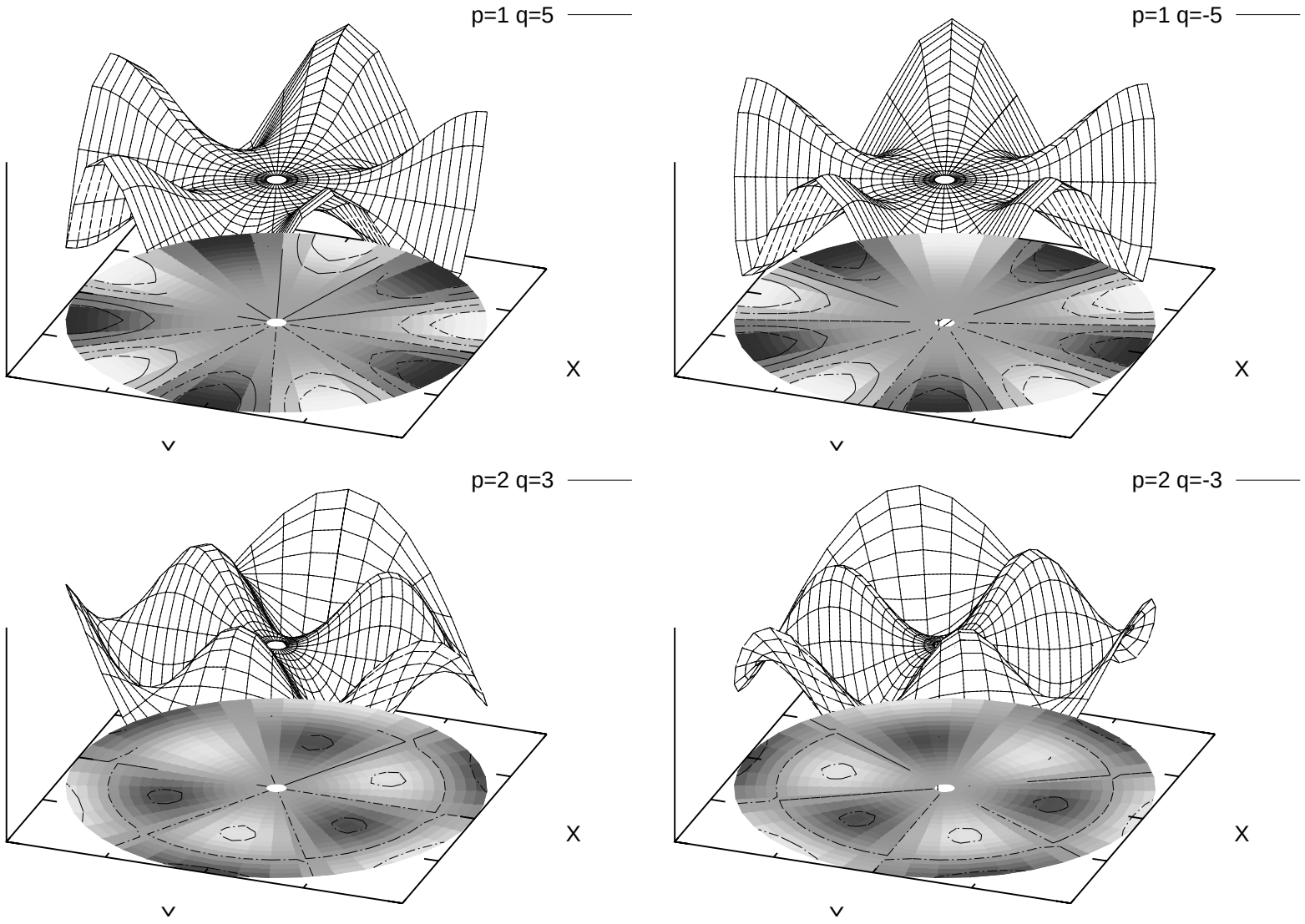
Snapshot Simulation.

KL Modes 12-14.



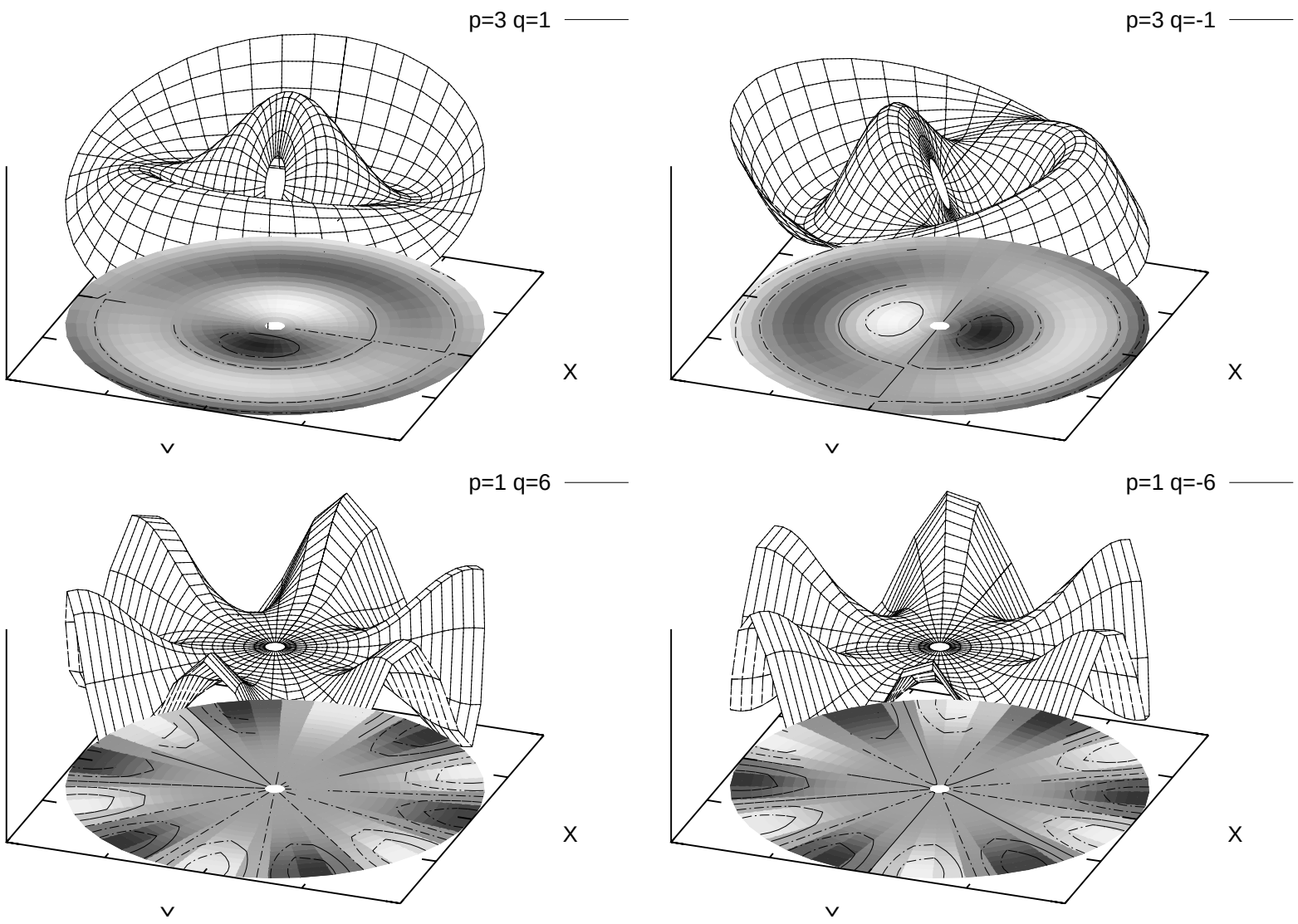
Snapshot Simulation.

KL Modes 15-18.



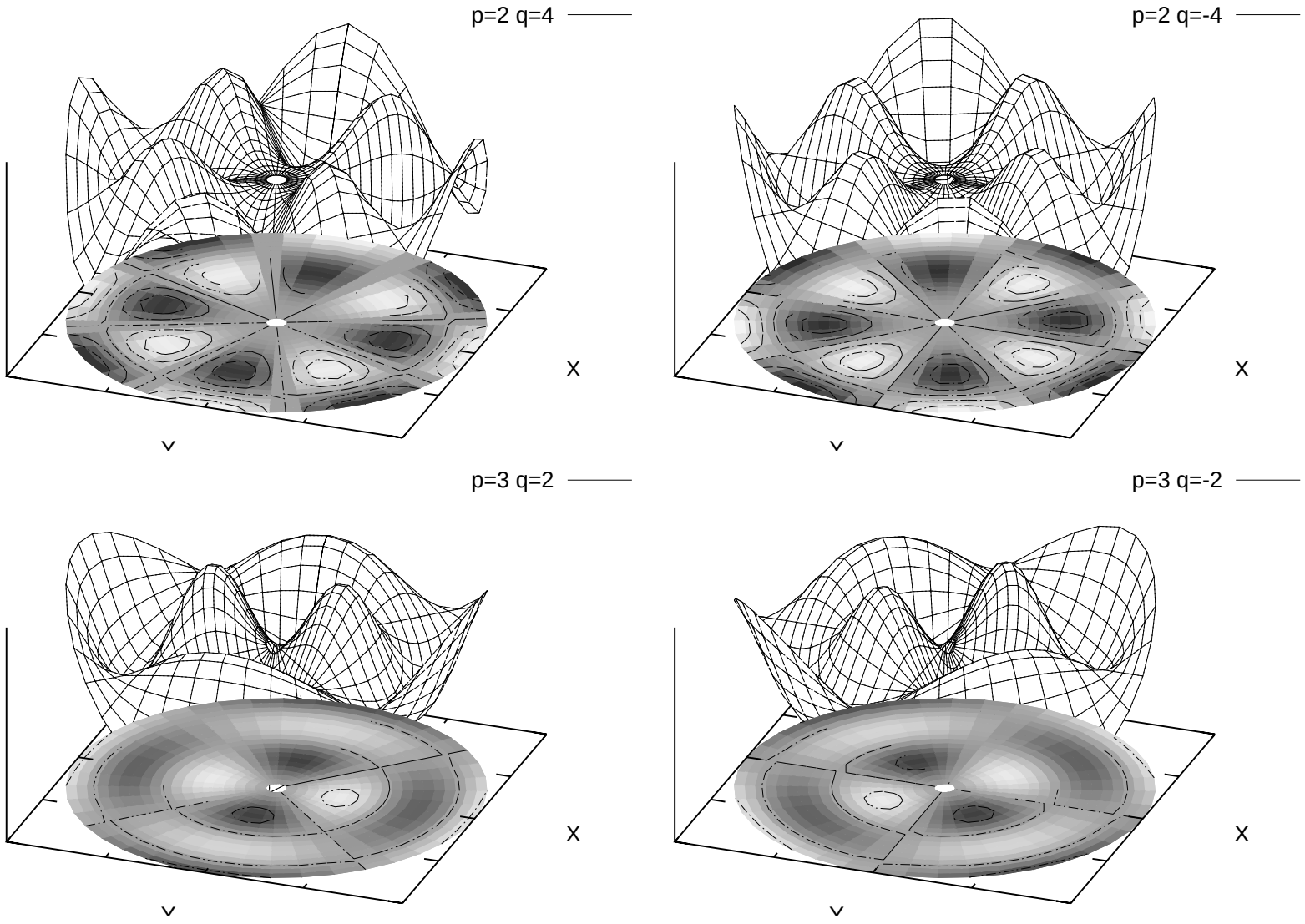
Snapshot Simulation.

KL Modes 19-22.



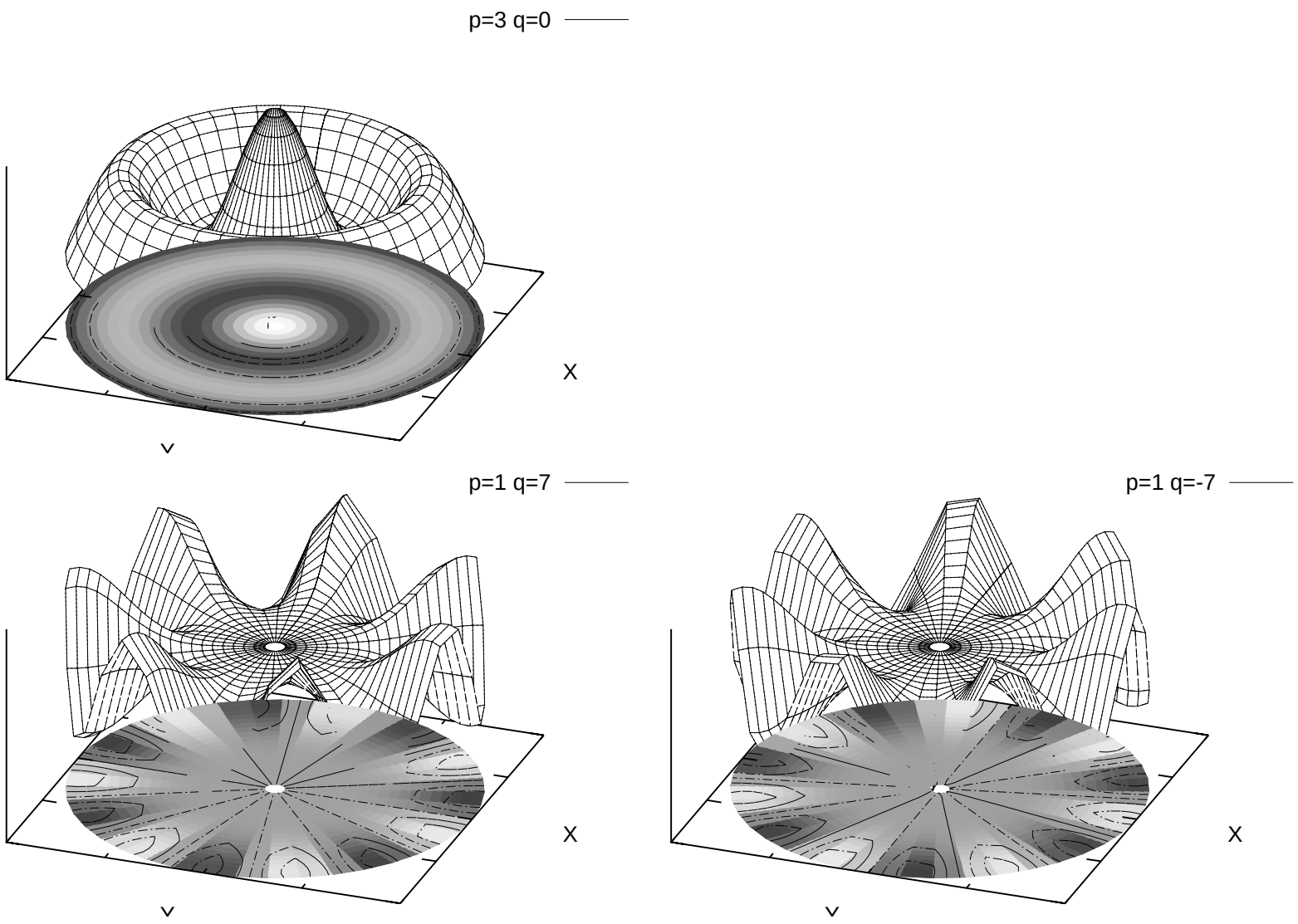
Snapshot Simulation.

KL Modes 23-26.



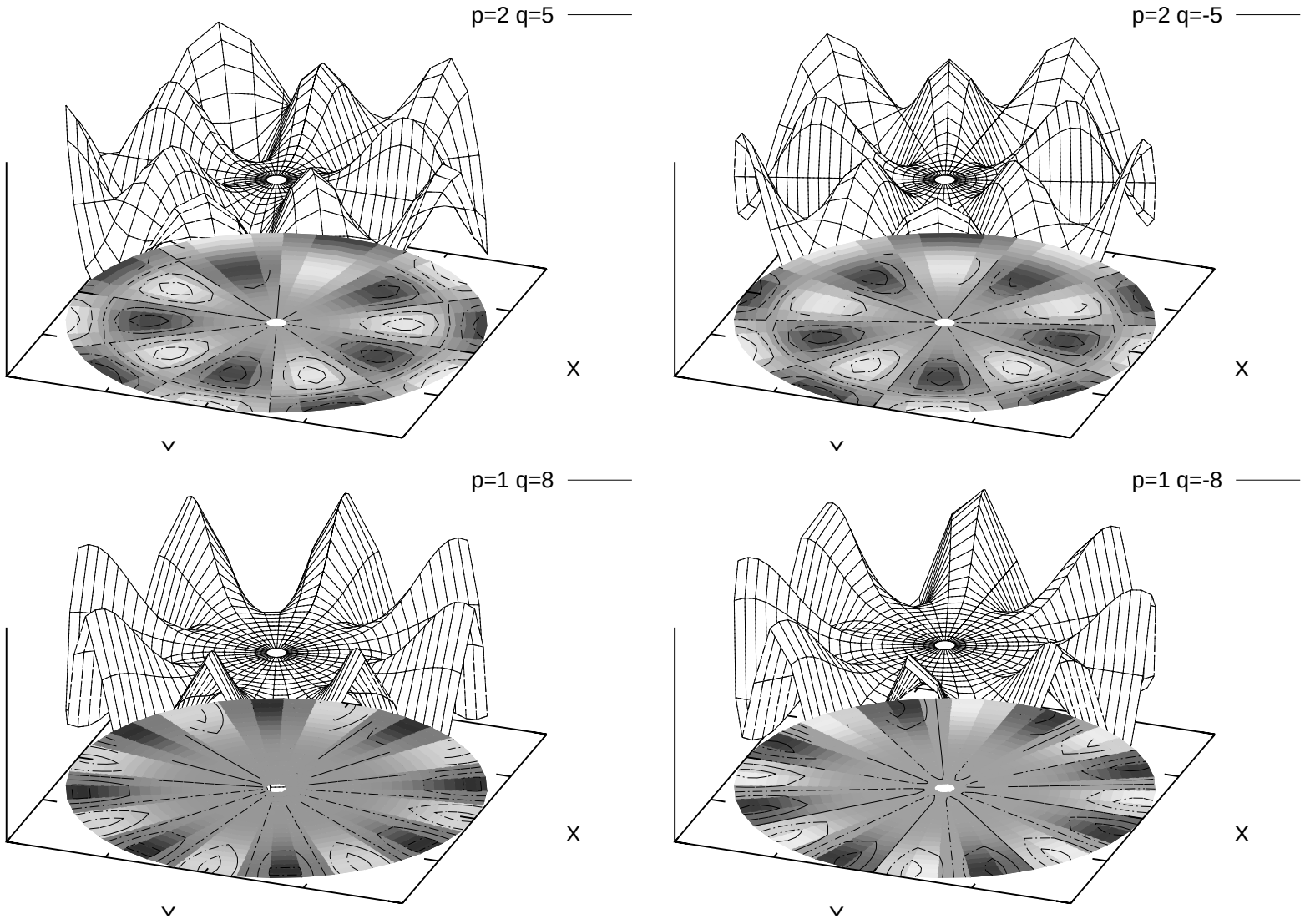
Snapshot Simulation.

KL Modes 27-29.



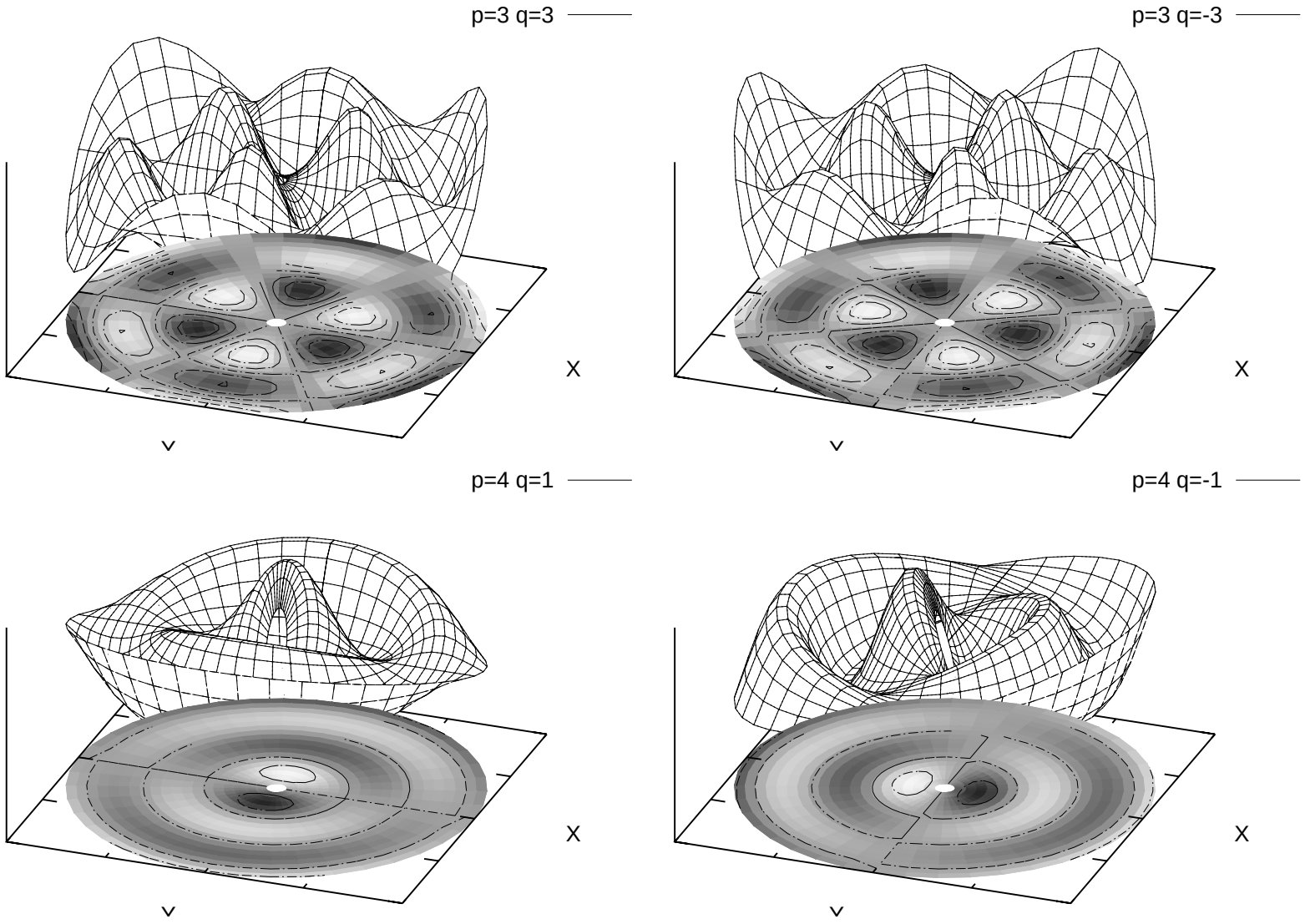
Snapshot Simulation.

KL Modes 30-33.



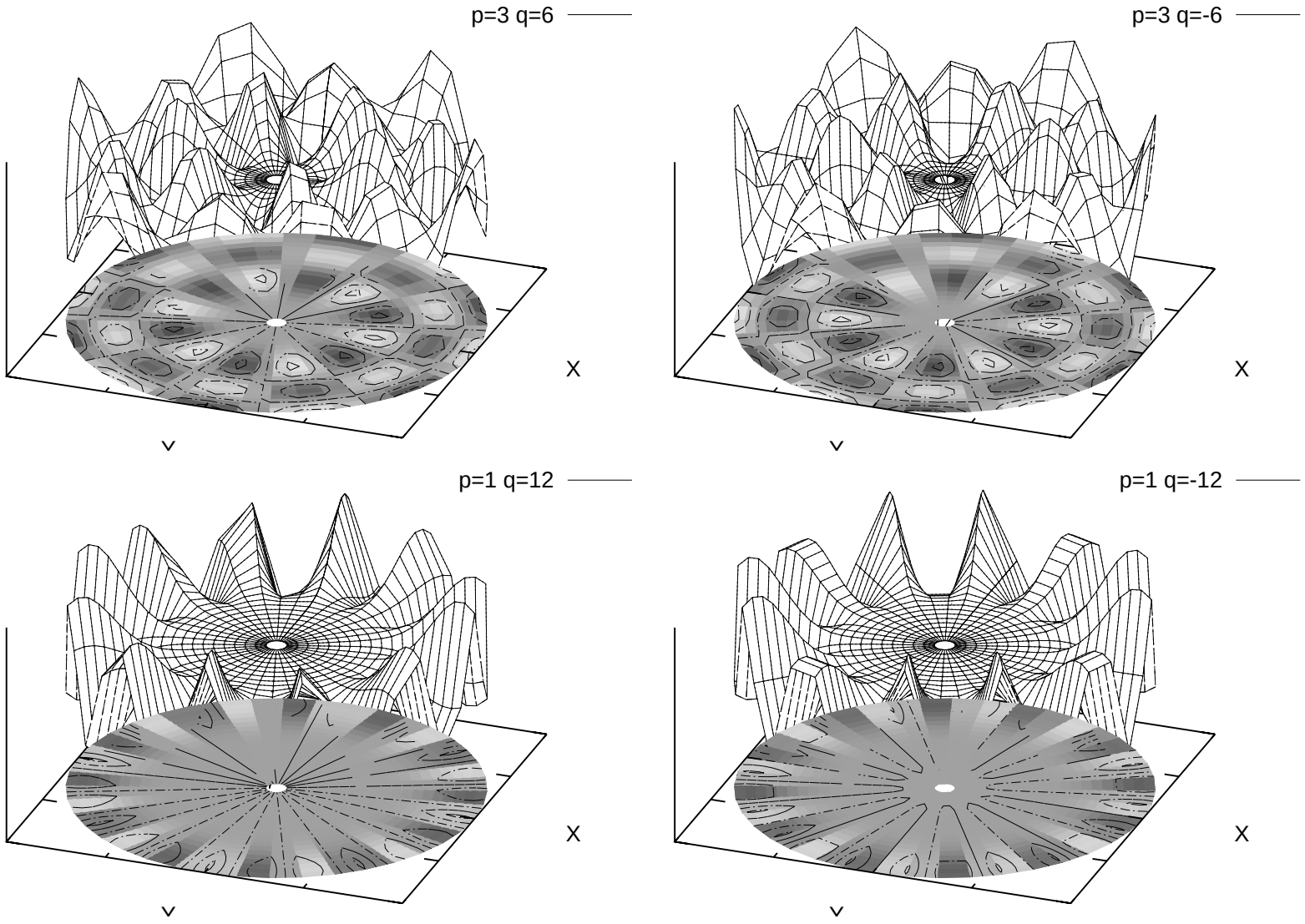
Snapshot Simulation.

KL Modes 34-37.



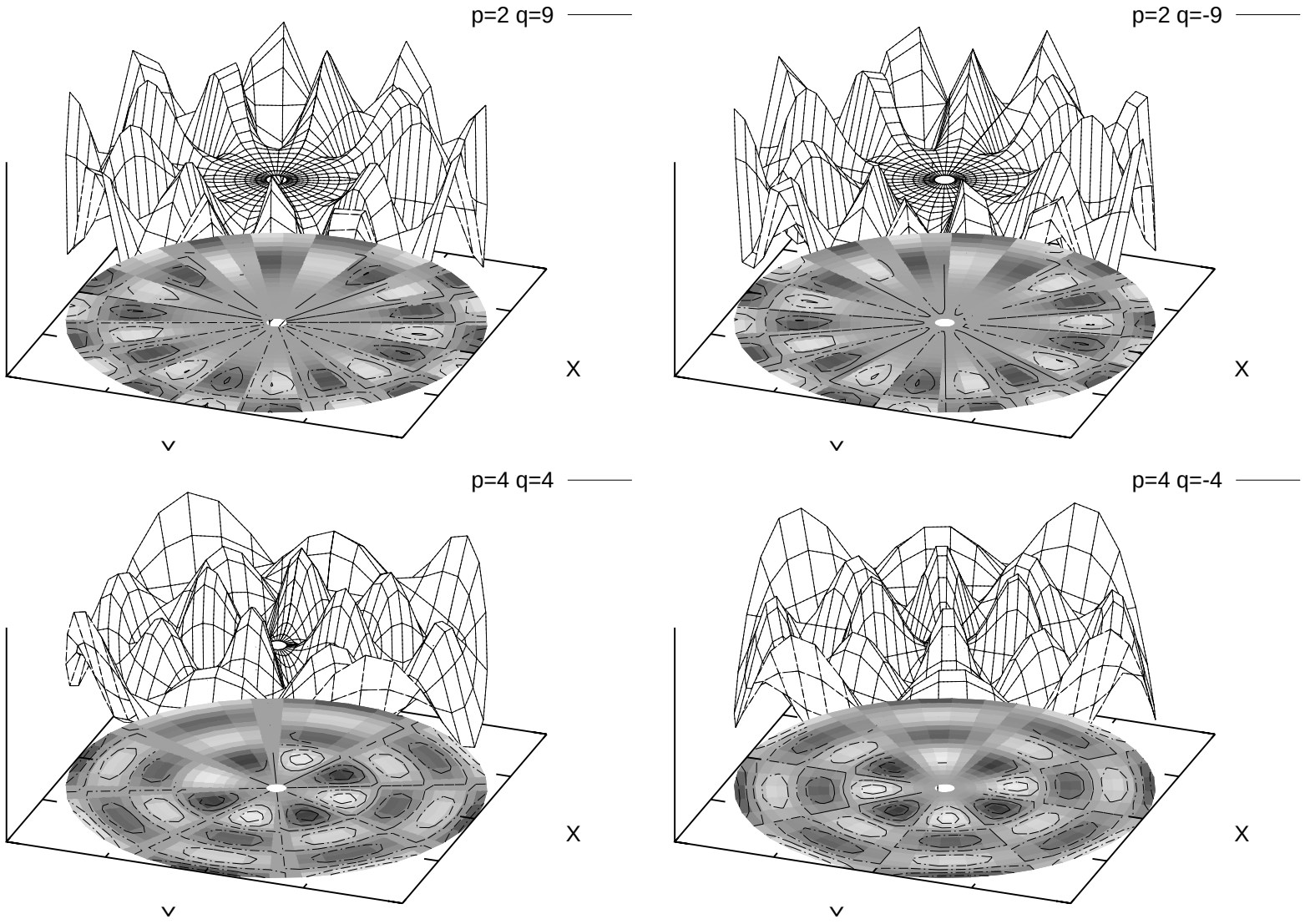
Snapshot Simulation.

KL Modes 61-64.



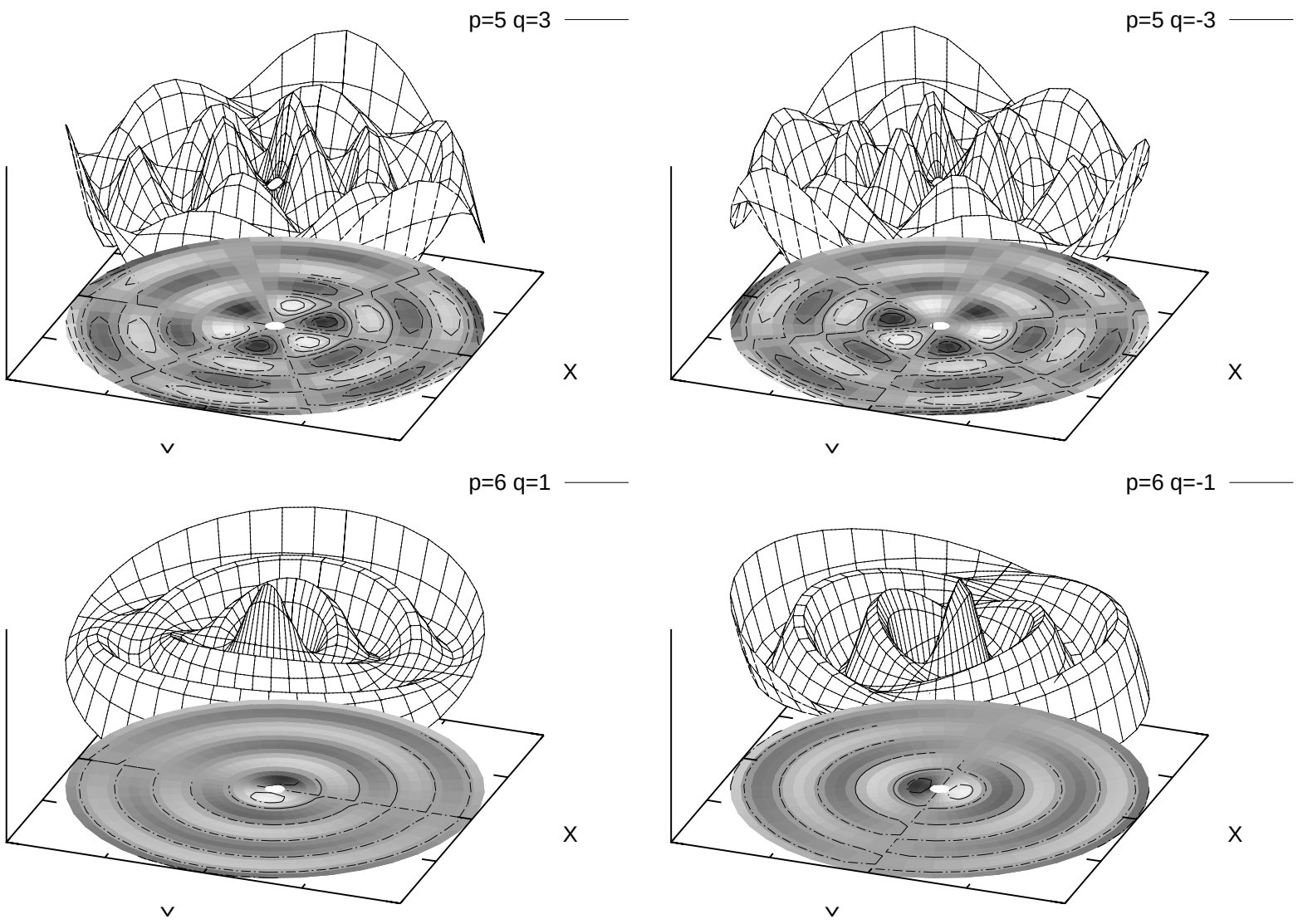
Snapshot Simulation.

KL Modes 65-68.



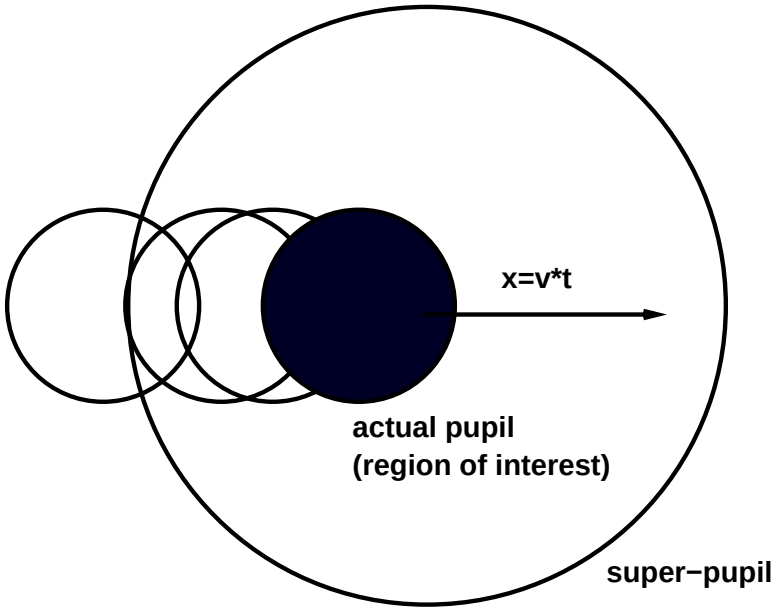
Snapshot Simulation.

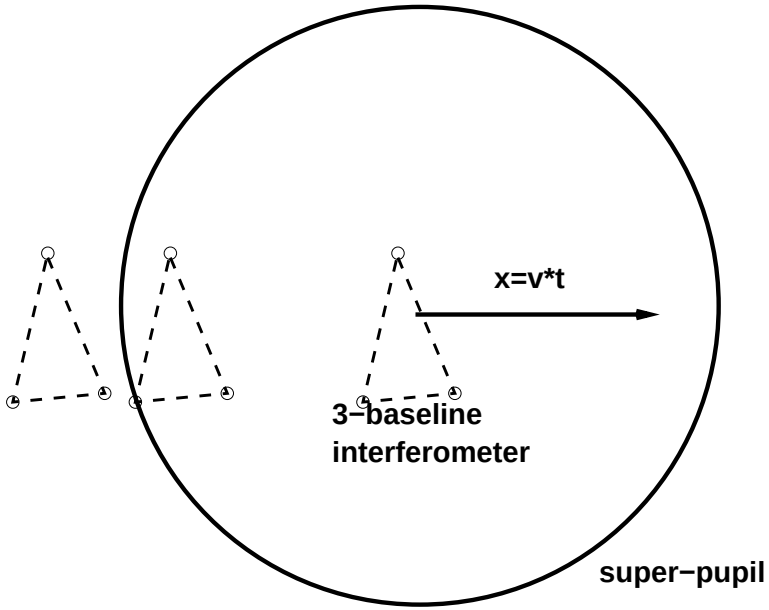
KL Modes 80-83.



Taylor Screen.

Brute force: subsampling super-pupils.





Simulation of animated phase screens with wind velocity v over time coordinate t:

1. Generate a static super-pupil of diameter vt (wasteful: mode set size grows $\sim$ diameter)
2. drag a circular pupil of the actual size across

Taylor Screen.

Alternative: De-centering, Mode Coupling [Mathar astro-ph/0709.0457].

Keep static modes $K_j(\vec{x})$ and work with time-dependent expansion coefficients $\beta(t)$:

$$\varphi(\vec{x}, t) = \varphi(\vec{x} - \vec{v}t, 0) \equiv \sum_{\text{modes } j} \beta_j(t)\lambda_j K_j(\vec{x}).$$

$$\varphi(\vec{k}, \omega) \propto \delta(\omega - \vec{k} \cdot \vec{v}).$$

Apply "frozen screen" operator

$$\partial_t \equiv -\vec{v} \cdot \vec{\nabla}$$

to this...

Taylor Screen.

First order differential equation.

...to obtain a system of first order homogeneous differential equations

$$\frac{d}{dt}\begin{pmatrix}\beta_1(t)\\ \beta_2(t)\\ \beta_3(t)\\ \cdots\end{pmatrix} \sim \begin{pmatrix} K_1\nabla K_1 & K_1\nabla K_2 & K_1\nabla K_3 & \cdots\\ K_2\nabla K_1 & K_2\nabla K_2 & K_2\nabla K_3 & \cdots\\ K_3\nabla K_1 & K_3\nabla K_2 & K_3\nabla K_3 & \cdots\\ \vdots & \vdots & & \ddots\end{pmatrix}\cdot\begin{pmatrix}\beta_1(t)\\ \beta_2(t)\\ \beta_3(t)\\ \cdots\end{pmatrix}$$

Antisymmetric gradient matrix, elements $\langle K_j \cdot \nabla K_i\rangle$. Sparse (similar to electric dipole selection rule of atomic transitions):

$$\left.\begin{matrix}\sin(q\theta)\\ \cos(q\theta)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\sin[(q\pm 1)\theta]\\ \cos[(q\pm 1)\theta]\end{matrix}\right. .$$

Missing randomness: an artifact of truncated order of matrix and vectors!

Taylor Screen.

Diagonalization, Dynamic Modes.

Standard treatment: diagonalize the gradient matrix. Work out the new, decoupled eigen-modes $K^{\mathrm{dyn}}(\vec{x})$ via the orthogonal transformation of the old $K_j(\vec{x})$ (once!).

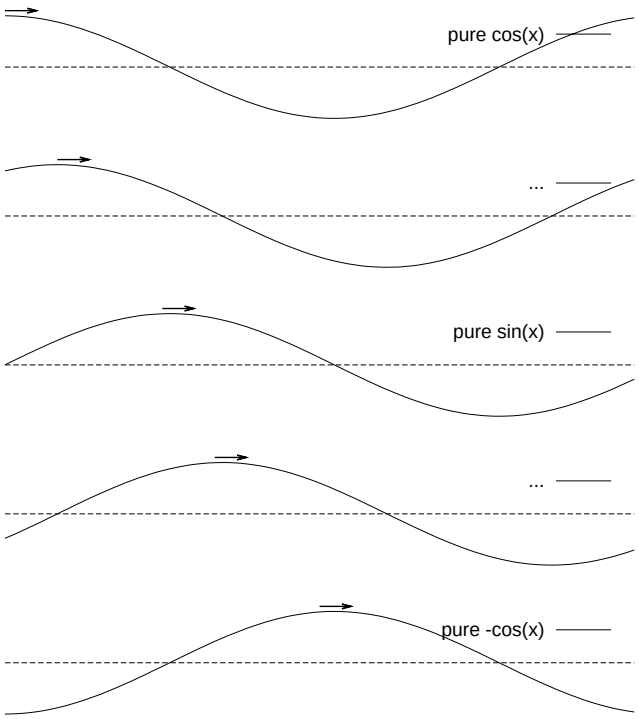
Numerically cheap: Expansion coefficients of these "traveling" modes run deterministically "on circles" in the complex plane,

$$\frac{d}{dt}\beta_j^{\mathrm{dyn}} = e^{\pm i\omega_j t}\beta_j^{\mathrm{dyn}}.$$

In pairs: ensures that a sine-component in $\vec{v}$-direction smoothly converts to a cosine-component after a time of $2\pi/\omega_j$ and again to a sine-component thereafter. Large ω_j (imaginary part of the eigenvalues of the gradient matrix) means small wiggles.

Taylor Screen.

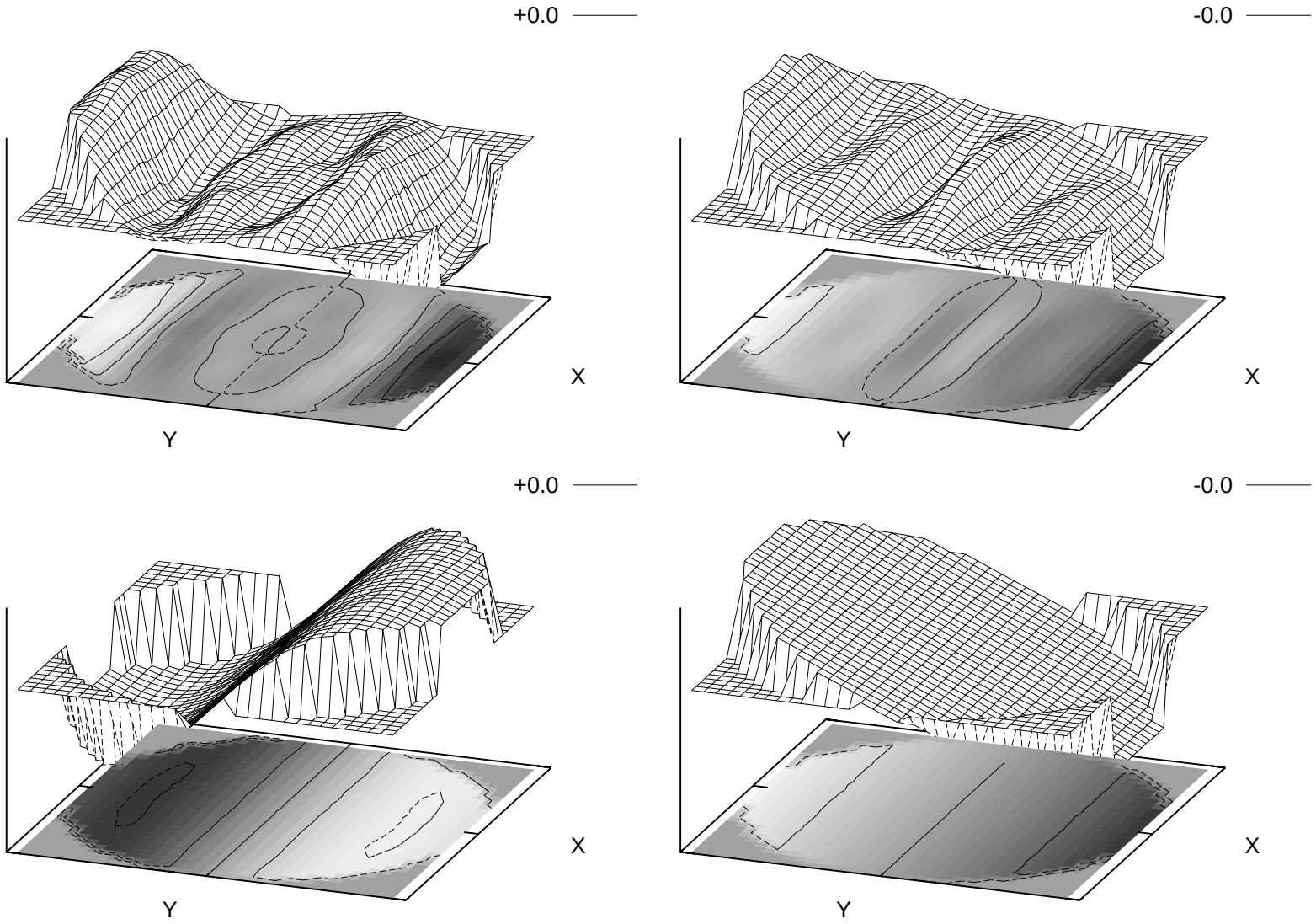
First order differential equation.



$$\cos(kx - \omega t) = cos(\omega t)\underbrace{\cos(kx)}_{\text{static}} + \sin(\omega t)\underbrace{\sin(kx)}_{\text{static shifted}90^\circ}$$

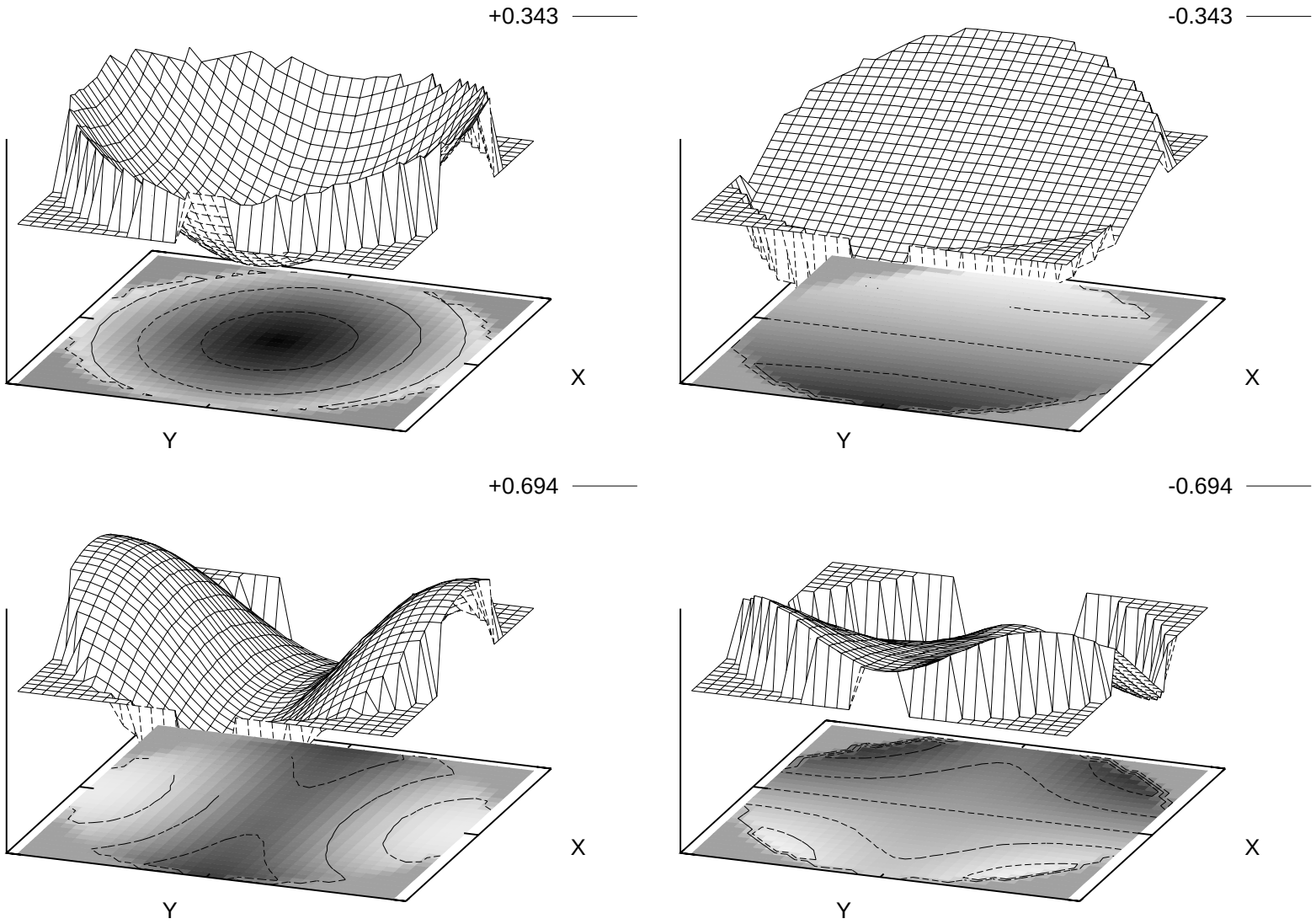
Taylor Screen.

Examples: almost non-periodic Dynamic Modes. Even or odd $\perp \vec{v}$.



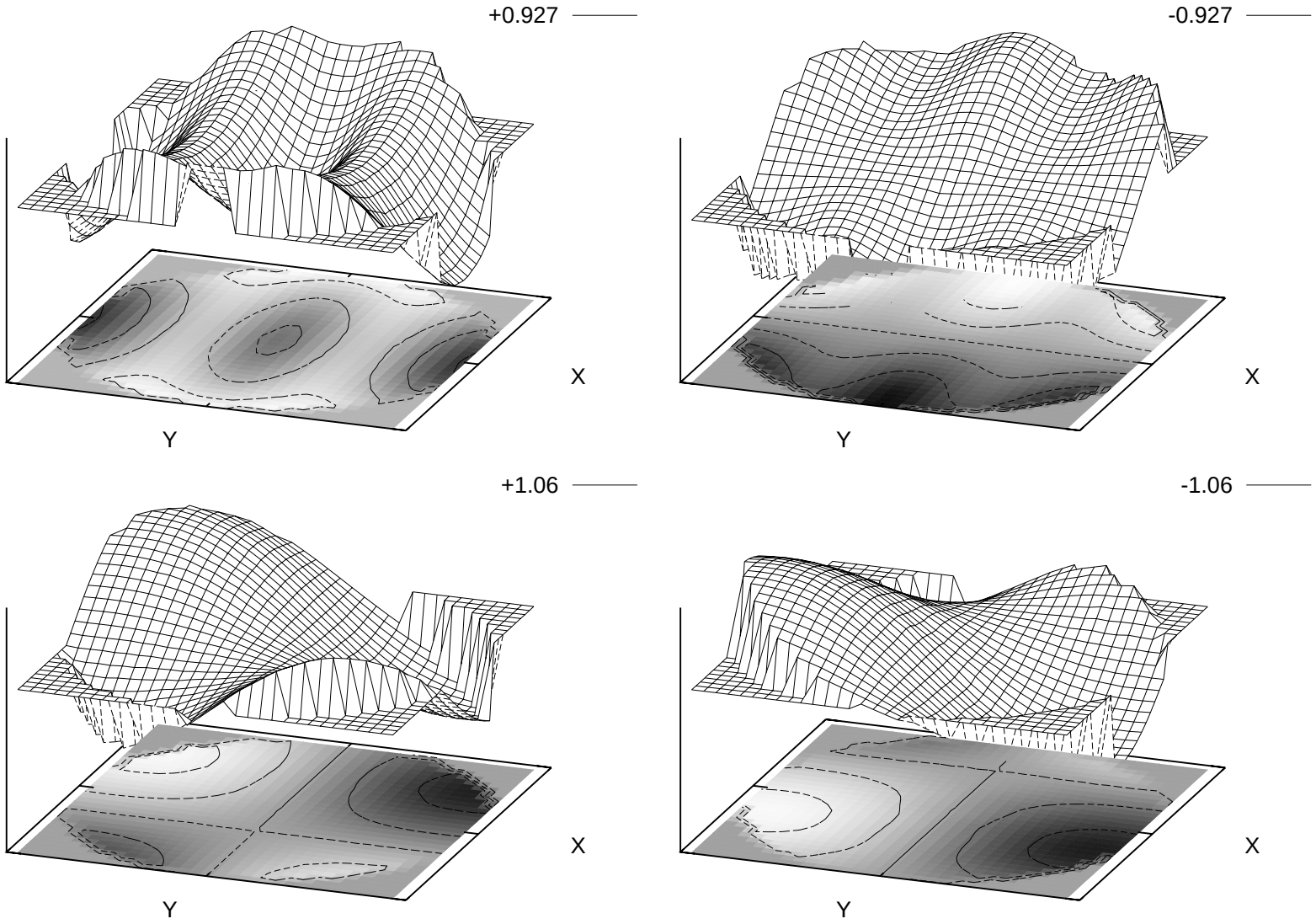
Taylor Screen.

Example: Pairs of Dynamic Modes. Large period $\parallel \vec{v}$.



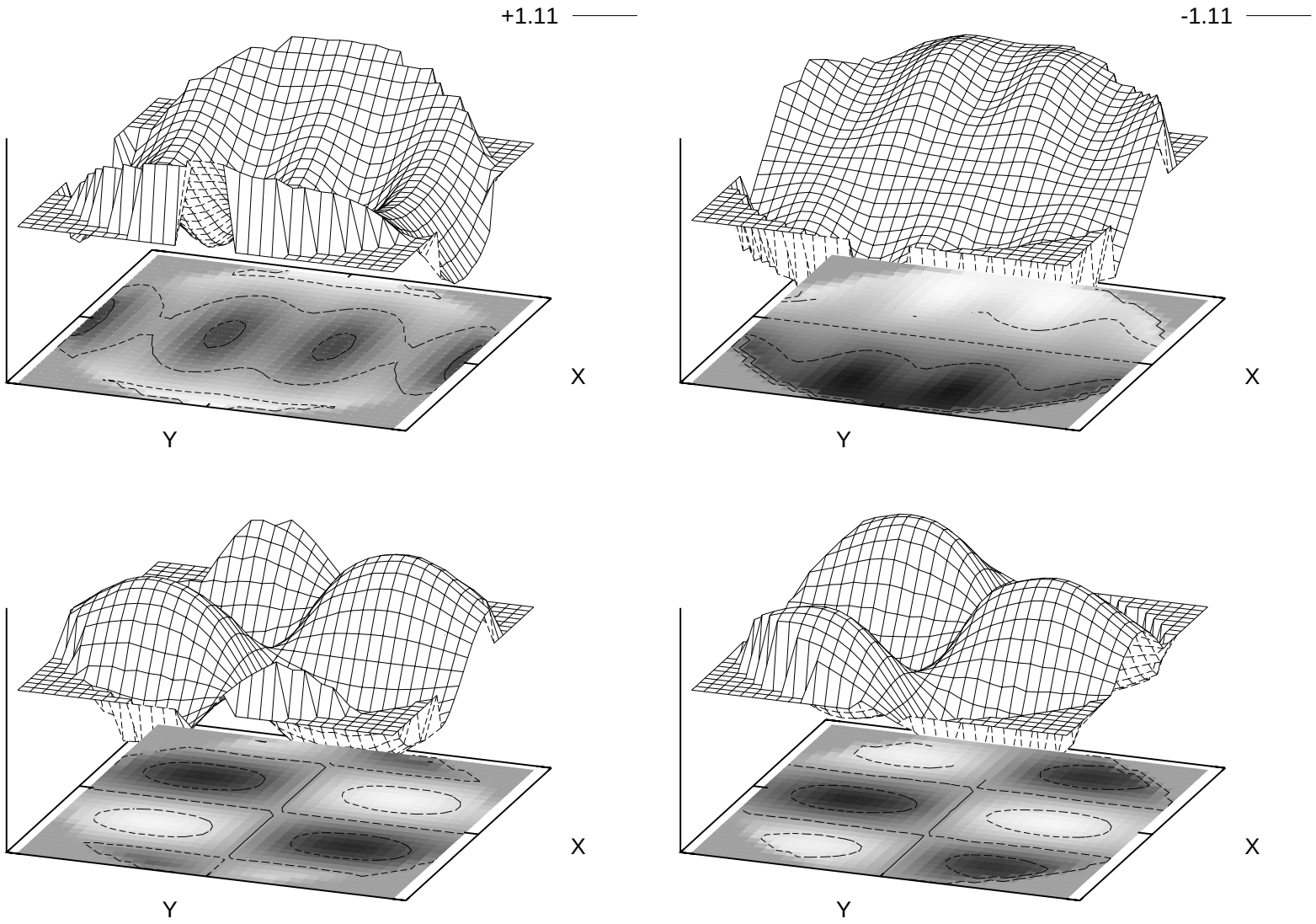
Taylor Screen.

Example: Pairs of Dynamic Modes. Periods near *D*.



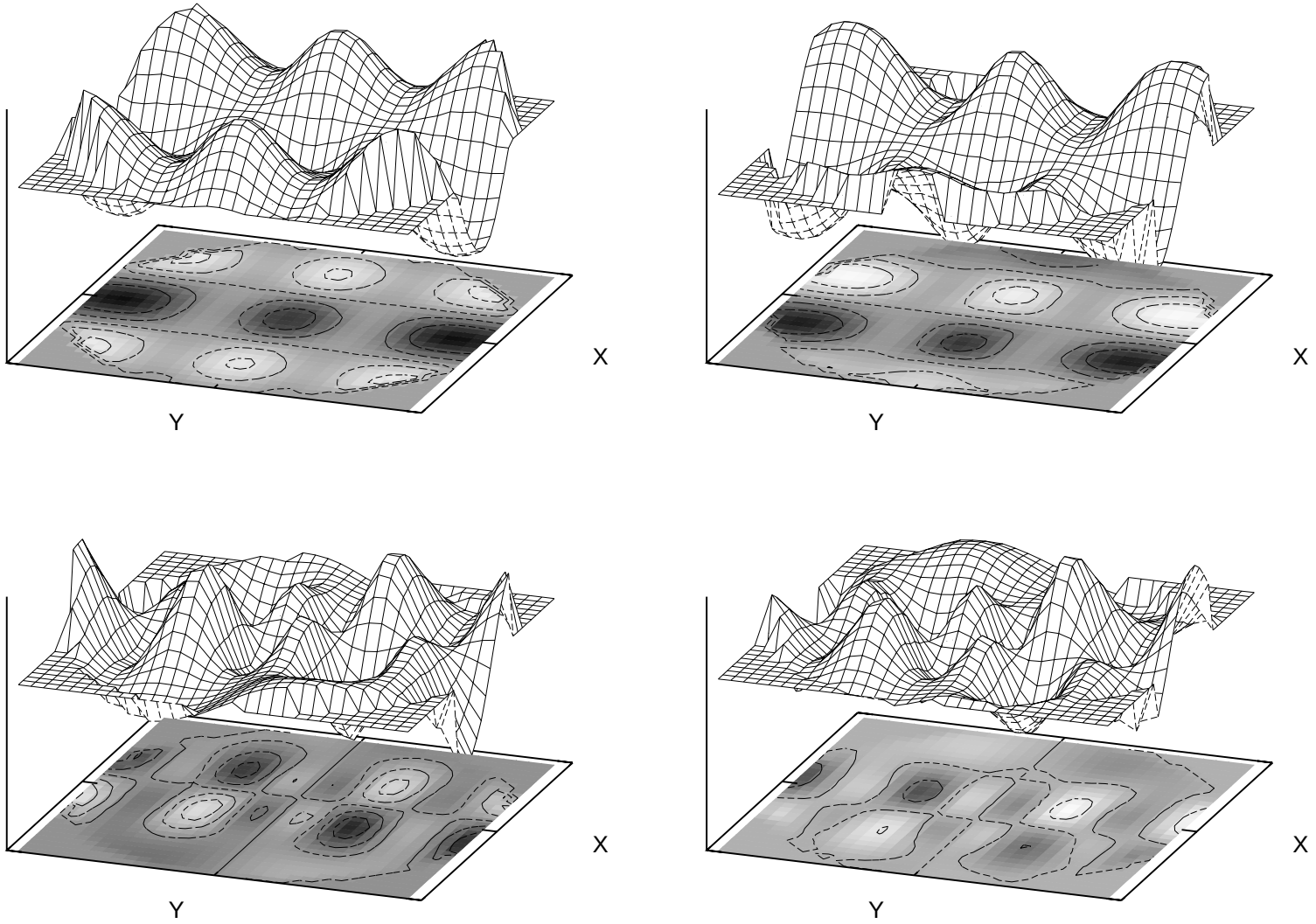
Taylor Screen.

Example: Pairs of Dynamic Modes. More wiggles $\perp \vec{v}$.



Taylor Screen.

Example: Pairs of Dynamic Modes. smaller periods $\parallel \vec{v}$.



Comparison of both Approaches.

Subsampling of static large screen vs. Mode Coupling.

Advantages

- Subsampling
  - simpler numerical implementation
  - works also for dis-connected regions of interest (interferometric arrays)
- Mode Coupling
  - phase screen movies of arbitrary duration
  - insight into equivalence transform: Breaking of rotational symmetry to 1-axis translational symmetry parallel to wind.

Summary.

The Karhunen-Loève integral equation defines statistically independent modes of any function given its correlation and a domain of sampling this correlation. This has been explored since the mid 70's for Kolmogorov phase statistics across circular pupils.

On time scales shorter than the speckle boiling time, statistical independence is replaced by the Taylor frozen screen assumption which drags the phase screen into some wind direction—predictably. I have outlined the mathematics of the associated first order differential equation (mode coupling), which morphs one superposition of modes into another on very long time scales.