Solving ODEs
Euler Method & RK2/4

Major: All Engineering Majors

Authors: Autar Kaw, Charlie Barker

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Euler Method

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Euler’s Method

\[ \frac{dy}{dx} = f(x, y), y(0) = y_0 \]

Slope \[= \frac{\text{Rise}}{\text{Run}}\]
\[= \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0) \]

\[ y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) \]
\[= y_0 + f(x_0, y_0)h \]

**Figure 1** Graphical interpretation of the first step of Euler’s method
Euler’s Method

\[ y_{i+1} = y_i + f(x_i, y_i)h \]

\[ h = x_{i+1} - x_i \]

**Figure 2.** General graphical interpretation of Euler’s method
How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \ y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \ y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$
Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

\[
\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8) \theta(0) = 1200K
\]

Find the temperature at \( t = 480 \) seconds using Euler’s method. Assume a step size of \( h = 240 \) seconds.
Solution

Step 1:

\[
\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right)
\]

\[
f(t, \theta) = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right)
\]

\[
\theta_{i+1} = \theta_i + f(t_i, \theta_i)h
\]

\[
\theta_1 = \theta_0 + f(t_0, \theta_0)h = 1200 + f(0,1200)240
\]

\[
= 1200 + \left( -2.2067 \times 10^{-12} \left( 1200^4 - 81 \times 10^8 \right) \right)240
\]

\[
= 1200 + (-4.5579)240
\]

\[
= 106.09 K
\]

\( \theta_1 \) is the approximate temperature at \( t = t_1 = t_0 + h = 0 + 240 = 240 \)

\[
\theta(240) \approx \theta_1 = 106.09 K
\]
Solution Cont

Step 2: For \( i = 1, \ t_1 = 240, \ \theta_1 = 106.09 \)

\[
\theta_2 = \theta_1 + f(t_1, \theta_1)h \\
= 106.09 + f(240, 106.09)240 \\
= 106.09 + \left( -2.2067 \times 10^{-12} \left( 106.09^4 - 81 \times 10^8 \right) \right)240 \\
= 106.09 + (0.017595)240 \\
= 110.32K
\]

\( \theta_2 \) is the approximate temperature at \( t = t_2 = t_1 + h = 240 + 240 = 480 \)

\[
\theta(480) \approx \theta_2 = 110.32K
\]
Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

\[ 0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \arctan(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282 \]

The solution to this nonlinear equation at \( t=480 \) seconds is

\[ \theta(480) = 647.57 K \]
Comparison of Exact and Numerical Solutions

Figure 3. Comparing exact and Euler’s method
Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, $h$

| Step, h | $\theta(480)$ | $E_t$   | $|\epsilon_t|\%$ |
|---------|----------------|---------|------------------|
| 480     | -987.8         | 1635.4  | 252.54           |
| 240     | 1              | 537.26  | 82.964           |
| 120     | 110.32         | 100.80  | 15.566           |
| 60      | 546.77         | 32.607  | 5.0352           |

$\theta(480) = 647.57K$ (exact)
Comparison with exact results

Figure 4. Comparison of Euler’s method with exact solution for different step sizes
Effects of step size on Euler’s Method

Figure 5. Effect of step size in Euler’s method.
Errors in Euler’s Method

It can be seen that Euler’s method has large errors. This can be illustrated using Taylor series.

\[ y_{i+1} = y_i + \frac{dy}{dx}_{x_i,y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2y}{dx^2}_{x_i,y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3y}{dx^3}_{x_i,y_i} (x_{i+1} - x_i)^3 + \ldots \]

\[ y_{i+1} = y_i + f(x_i, y_i) (x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i) (x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i) (x_{i+1} - x_i)^3 + \ldots \]

As you can see the first two terms of the Taylor series

\[ y_{i+1} = y_i + f(x_i, y_i)h \]

are the Euler’s method.

The true error in the approximation is given by

\[ E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots \]

\[ E_t \propto h^2 \]
Runge 2\textsuperscript{nd} Order Method

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Runge-Kutta 2\textsuperscript{nd} Order Method

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Runge-Kutta 2\textsuperscript{nd} Order Method

For \[
\frac{dy}{dx} = f(x, y), \quad y(0) = y_0
\]

Runge Kutta 2nd order method is given by

\[y_{i+1} = y_i + (a_1k_1 + a_2k_2)h\]

where

\[k_1 = f(x_i, y_i)\]
\[k_2 = f(x_i + p_1h, y_i + q_1k_1h)\]
Heun’s Method

Heun’s method

Here $a_2=1/2$ is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1h)$$

Figure 1  Runge-Kutta 2nd order method (Heun’s method)
Here $a_2 = 1$ is chosen, giving

\[ a_1 = 0 \]
\[ p_1 = \frac{1}{2} \]
\[ q_{11} = \frac{1}{2} \]

resulting in

\[ y_{i+1} = y_i + k_2 h \]

where

\[ k_1 = f(x_i, y_i) \]
\[ k_2 = f\left(x_i + \frac{1}{2} h, y_i + \frac{1}{2} k_1 h\right) \]
Ralston’s Method

Here \( a_2 = \frac{2}{3} \) is chosen, giving

\[
\begin{align*}
    a_1 & = \frac{1}{3} \\
    p_1 & = \frac{3}{4} \\
    q_{11} & = \frac{3}{4}
\end{align*}
\]

resulting in

\[
y_{i+1} = y_i + \left( \frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h
\]

where

\[
\begin{align*}
    k_1 & = f \left( x_i, y_i \right) \\
    k_2 & = f \left( x_i + \frac{3}{4} h, y_i + \frac{3}{4} k_1 h \right)
\end{align*}
\]
How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

\[
\frac{dy}{dx} = f(x, y)
\]

**Example**

\[
\frac{dy}{dx} + 2y = 1.3e^{-x}, \ y(0) = 5
\]

is rewritten as

\[
\frac{dy}{dx} = 1.3e^{-x} - 2y, \ y(0) = 5
\]

In this case

\[
f(x, y) = 1.3e^{-x} - 2y
\]
Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

\[ \frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \theta(0) = 1200K \]

Find the temperature at \( t = 480 \) seconds using Heun’s method. Assume a step size of \( h = 240 \) seconds.

\[ \frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \]

\[ f(t, \theta) = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \]

\[ \theta_{i+1} = \theta_i + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \]
Solution

Step 1: \[ i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K \]

\[ k_1 = f(t_0, \theta_0) \]
\[ = f(0, 1200) \]
\[ = -2.2067 \times 10^{-12} \left( 1200^4 - 81 \times 10^8 \right) \]
\[ = -4.5579 \]

\[ k_2 = f(t_0 + h, \theta_0 + k_1 h) \]
\[ = f(0 + 240, 1200 + (-4.5579) \times 240) \]
\[ = f(240, 106.09) \]
\[ = -2.2067 \times 10^{-12} \left( 106.09^4 - 81 \times 10^8 \right) \]
\[ = 0.017595 \]

\[ \theta_1 = \theta_0 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \]
\[ = 1200 + \left( \frac{1}{2} (-4.5579) + \frac{1}{2} (0.017595) \right) \times 240 \]
\[ = 1200 + (-2.2702) \times 240 \]
\[ = 655.16K \]
Solution Cont

Step 2: \( i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K \)

\[
\begin{align*}
    k_1 &= f(t_1, \theta_1) \\
     &= f(240, 655.16) \\
     &= -2.2067 \times 10^{-12} \left( 655.16^4 - 81 \times 10^8 \right) \\
     &= -0.38869
\end{align*}
\]

\[
\begin{align*}
    k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\
     &= f(240 + 240, 655.16 + (-0.38869)240) \\
     &= f(480, 561.87) \\
     &= -2.2067 \times 10^{-12} \left( 561.87^4 - 81 \times 10^8 \right) \\
     &= -0.20206
\end{align*}
\]

\[
\begin{align*}
    \theta_2 &= \theta_1 + \left( \frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h \\
             &= 655.16 + \left( \frac{1}{2} (-0.38869) + \frac{1}{2} (-0.20206) \right) 240 \\
             &= 655.16 + (-0.29538) 240 \\
             &= 584.27K
\end{align*}
\]
The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

\[
0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282
\]

The solution to this nonlinear equation at \( t=480 \) seconds is

\[
\theta (480) = 647.57K
\]
Comparison with exact results

Figure 2. Heun’s method results for different step sizes

- Exact
- h=120
- h=240
- h=480

Temperature, $\theta$(K)

Time, t(sec)
Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

| Step size, h | θ(480)   | $E_t$    | $|\epsilon_t|$% |
|--------------|----------|----------|-----------------|
| 480          | −393.87  | 1041.4   | 160.82          |
| 240          | 584.27   | 63.304   | 9.7756          |
| 120          | 651.35   | −3.7762  | 0.58313         |
| 60           | 649.91   | −2.3406  | 0.36145         |

$\theta (480) = 647.57 K$ (exact)
Effects of step size on Heun’s Method

Figure 3. Effect of step size in Heun’s method
Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

<table>
<thead>
<tr>
<th>Step size, $h$</th>
<th>$\theta(480)$</th>
<th>Euler</th>
<th>Heun</th>
<th>Midpoint</th>
<th>Ralston</th>
</tr>
</thead>
<tbody>
<tr>
<td>480</td>
<td>$-987.84$</td>
<td>$-393.87$</td>
<td>1208.4</td>
<td>449.78</td>
<td></td>
</tr>
<tr>
<td>240</td>
<td>110.32</td>
<td>584.27</td>
<td>976.87</td>
<td>690.01</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>546.77</td>
<td>651.35</td>
<td>690.20</td>
<td>667.71</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>614.97</td>
<td>649.91</td>
<td>654.85</td>
<td>652.25</td>
<td></td>
</tr>
</tbody>
</table>

$\theta(480) = 647.57K$ (exact)
**Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods**

**Table 2.** Comparison of Euler and the Runge-Kutta methods

| Step size, $h$ | $|\varepsilon_t|\%$ Euler | $|\varepsilon_t|\%$ Heun | $|\varepsilon_t|\%$ Midpoint | $|\varepsilon_t|\%$ Ralston |
|---------------|-------------------------------|----------------|----------------|-------------------|
| 480           | 252.54                        | 160.82         | 86.612         | 30.544            |
| 240           | 82.964                        | 9.7756         | 50.851         | 6.5537            |
| 120           | 15.566                        | 0.58313        | 6.5823         | 3.1092            |
| 60            | 5.0352                        | 0.36145        | 1.1239         | 0.7229            |
| 30            | 2.2864                        | 0.097625       | 0.22353        | 9                 |

$\theta(480) = 647.57K$ (exact)
Comparison of Euler and Runge-Kutta 2\textsuperscript{nd} Order Methods

\textbf{Figure 4.} Comparison of Euler and Runge Kutta 2\textsuperscript{nd} order methods with exact results.
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html
Runge 4\textsuperscript{th} Order Method

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Runge-Kutta 4\textsuperscript{th} Order Method

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Runge-Kutta 4\textsuperscript{th} Order Method

For \[ \frac{dy}{dx} = f(x, y), \ y(0) = y_0 \]

Runge Kutta 4\textsuperscript{th} order method is given by

\[ y_{i+1} = y_i + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) h \]

where

\[ k_1 = f(x_i, y_i) \]

\[ k_2 = f\left( x_i + \frac{1}{2} h, y_i + \frac{1}{2} k_1 h \right) \]

\[ k_3 = f\left( x_i + \frac{1}{2} h, y_i + \frac{1}{2} k_2 h \right) \]

\[ k_4 = f\left( x_i + h, y_i + k_3 h \right) \]
How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

\[ \frac{dy}{dx} = f(x, y) \]

**Example**

\[ \frac{dy}{dx} + 2y = 1.3e^{-x}, \quad y(0) = 5 \]

is rewritten as

\[ \frac{dy}{dx} = 1.3e^{-x} - 2y, \quad y(0) = 5 \]

In this case

\[ f(x, y) = 1.3e^{-x} - 2y \]
Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

\[ \frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \theta(0) = 1200K \]

Find the temperature at \( t = 480 \) seconds using Runge-Kutta 4\(^{th}\) order method.

Assume a step size of \( h = 240 \) seconds.

\[ \frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \]

\[ f(t, \theta) = -2.2067 \times 10^{-12} \left( \theta^4 - 81 \times 10^8 \right) \]

\[ \theta_{i+1} = \theta_i + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) h \]
Solution

Step 1: \( i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200 \)

\[ k_1 = f(t_0, \theta_0) = f(0, 1200) = -2.2067 \times 10^{-12} \left( 1200^4 - 81 \times 10^8 \right) = -4.5579 \]

\[ k_2 = f \left( t_0 + \frac{1}{2} h, \theta_0 + \frac{1}{2} k_1 h \right) = f \left( 0 + \frac{1}{2} (240) 1200 + \frac{1}{2} (-4.5579) 240 \right) \]

\[ = f(120, 653.05) = -2.2067 \times 10^{-12} \left( 653.05^4 - 81 \times 10^8 \right) = -0.38347 \]

\[ k_3 = f \left( t_0 + \frac{1}{2} h, \theta_0 + \frac{1}{2} k_2 h \right) = f \left( 0 + \frac{1}{2} (240) 1200 + \frac{1}{2} (-0.38347) 240 \right) \]

\[ = f(120, 1154.0) = 2.2067 \times 10^{-12} \left( 1154.0^4 - 81 \times 10^8 \right) = -3.8954 \]

\[ k_4 = f \left( t_0 + h, \theta_0 + k_3 h \right) = f \left( 0 + (240) 1200 + (-3.984) 240 \right) \]

\[ = f(240, 265.10) = 2.2067 \times 10^{-12} \left( 265.10^4 - 81 \times 10^8 \right) = 0.0069750 \]
Solution Cont

\[ \theta_1 = \theta_0 + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) h \]

\[ = 1200 + \frac{1}{6} \left( -4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750) \right) 240 \]

\[ = 1200 + \frac{1}{6} (-2.1848) 240 \]

\[ = 675.65 K \]

\[ \theta_1 \] is the approximate temperature at

\[ t = t_1 = t_0 + h = 0 + 240 = 240 \]

\[ \theta(240) \approx \theta_1 = 675.65 K \]
Solution Cont

Step 2: \( i = 1, t_1 = 240, \theta_1 = 675.65K \)

\[
k_1 = f(t_1, \theta_1) = f(240, 675.65) = -2.2067 \times 10^{-12} \left( 675.65^4 - 81 \times 10^8 \right) = -0.44199
\]

\[
k_2 = f\left( t_1 + \frac{1}{2} h, \theta_1 + \frac{1}{2} k_1 h \right) = f\left( 240 + \frac{1}{2} (240), 675.65 + \frac{1}{2} (-0.44199)240 \right)
\]
\[
= f(360, 622.61) = -2.2067 \times 10^{-12} \left( 622.61^4 - 81 \times 10^8 \right) = -0.31372
\]

\[
k_3 = f\left( t_1 + \frac{1}{2} h, \theta_1 + \frac{1}{2} k_2 h \right) = f\left( 240 + \frac{1}{2} (240), 675.65 + \frac{1}{2} (-0.31372)240 \right)
\]
\[
= f(360, 638.00) = 2.2067 \times 10^{-12} \left( 638.00^4 - 81 \times 10^8 \right) = -0.34775
\]

\[
k_4 = f(t_1 + h, \theta_1 + k_3 h) = f(240 + (240), 675.65 + (-0.34775)240)
\]
\[
= f(480, 592.19) = 2.2067 \times 10^{-12} \left( 592.19^4 - 81 \times 10^8 \right) = -0.25351
\]
Solution Cont

\[ \theta_2 = \theta_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) h \]

\[ = 675.65 + \frac{1}{6} (-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351)) 240 \]

\[ = 675.65 + \frac{1}{6} (-2.0184) 240 \]

\[ = 594.91 K \]

\( \theta_2 \) is the approximate temperature at

\[ t_2 = t_1 + h = 240 + 240 = 480 \]

\[ \theta (480) \approx \theta_2 = 594.91 K \]
The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

\[ 0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282 \]

The solution to this nonlinear equation at \( t=480 \) seconds is

\[ \theta(480) = 647.57 K \]
Comparison with exact results

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution
## Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

| Step size, h | \( \theta \) (480) | \( E_t \) | \( |\epsilon_t| \) % |
|--------------|---------------------|----------|---------------------|
| 480          | -90.278             | 737.85   | 113.94              |
| 240          | 594.91              | 52.660   | 8.1319              |
| 120          | 646.16              | 1.4122   | 0.21807             |
| 60           | 647.54              | 0.033626 | 0.0051926           |
| 30           | 647.57              | 0.00086900 | 0.00013419          |

\[ \theta (480) = 647.57 \, K \quad \text{(exact)} \]
Effects of step size on Runge-Kutta 4th Order Method

Figure 2. Effect of step size in Runge-Kutta 4th order method
Comparison of Euler and Runge-Kutta Methods

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.
THE END