10. II. Describing Space: Isotropic Curved Spaces

Starting point:

What is the universe expanding into?

- The observable universe is a lower dimensional sub-space expanding within a higher dimensional space.

OR

- We can describe the expanding 3D universe without reference to higher dimensions (has proven more useful prescription).

Note: Here, we restrict ourselves to the macrosopic description of curved space; all issues of quantum gravity (string theory) will be left out.

II.1 A brief reminder about metrics

Example 1:

Static, Euclidean space can be fully described by its metric, describing the distance *dl* between two nearby points:

$$dl^2 = dx^2 + dy^2 + dz^2$$

In general, a metric is written in tensor form:

$$dl^2 = g_{\mu\nu} \cdot dx^{\mu}\, dx^{\nu}$$

with $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for the example above.

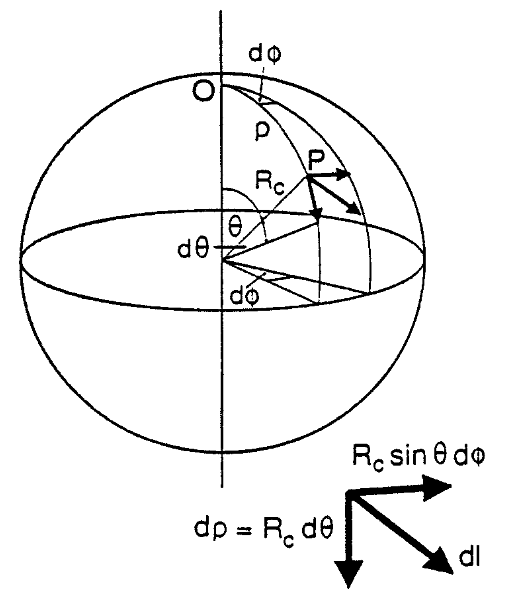
Note: For static, Euclidean space $g_{\mu\nu}$ does **not** depend on:

- the co-ordinates dx^{μ}
- time

but this is special!

Example 2: Surface of a sphere

Simplest example of an isotropic, curved, two-dimensional space.
Note: There is no preferred direction, even if we pick a polar co-ordinate system.



Metric: $$dl^2 = R_c^2 \, d\vartheta^2 + R_c^2 \sin^2 \vartheta \, d\varphi^2$$

Define $\hat{\rho}$ as the shortest distance (along the "great circle") between two points, e.g. O and P ("geodesic distance"):

$$\Rightarrow \quad \vartheta = \frac{\hat{\rho}}{R_c} \quad \text{and} \quad dl^2 = d\hat{\rho}^2 + R_c^2 \sin^2(\hat{\rho}/R)\, d\varphi^2$$

Now define:

$$x \equiv R_c \sin\left(\frac{\hat{\rho}}{R_c}\right) \quad \Rightarrow \quad dx^2 = \left[1 - \sin^2\left(\frac{\hat{\rho}}{R_c}\right)\right] d\varphi^2$$

and define the curvature $\kappa \equiv \frac{1}{R_c^2}$

$$\Rightarrow d\hat{\rho}^2 = \frac{dx^2}{1-\kappa x^2}$$

$\Rightarrow dl^2 = dx^2 / (1-\kappa x^2) + x^2 d\varphi^2$
metric for the 2-sphere

Note: $d\rho = \frac{dl}{x} \Rightarrow x$ measures the "angular diameter distance" on a sphere (as in Euclidean space)!

Note:

- The metric

$$g = \begin{pmatrix} \frac{1}{1-\kappa x^2} & 0 \\ 0 & x^2 \end{pmatrix}$$

describes the isotropic 2D-sphere of curvature κ without reference to a 3rd dimension.

- Three qualitatively different cases:
  (1) $\kappa > 0 \Rightarrow R_c$ real: spherical space
  (2) $\kappa = 0 \Rightarrow R_c \rightarrow \infty$: flat, Euclidean space
  (3) $\kappa = 0 \Rightarrow R_c$ imaginary: hyperbolic space

- We can either make the radial part of the metric look Euclidean:

$$dl^2 = d\hat{\rho}^2 + R_c^2 \sin^2\left(\frac{\hat{\rho}}{R_c}\right) d\varphi^2$$

or the tangential part $dl^2 = \frac{dx^2}{1-\kappa x^2} + x^2 d\varphi^2$

- So far, static metric; no time dependence.

II.2 Robertson-Walker Metric

- Generalize the metric of the "two-sphere" to three dimensions:

$$dl^2 = d\hat{\rho}^2 + R_c^2 \sin^2\left(\frac{\hat{\rho}}{R_c}\right) \cdot \left[d^2\vartheta + \sin^2\vartheta \, d\varphi^2\right] \quad *$$

or

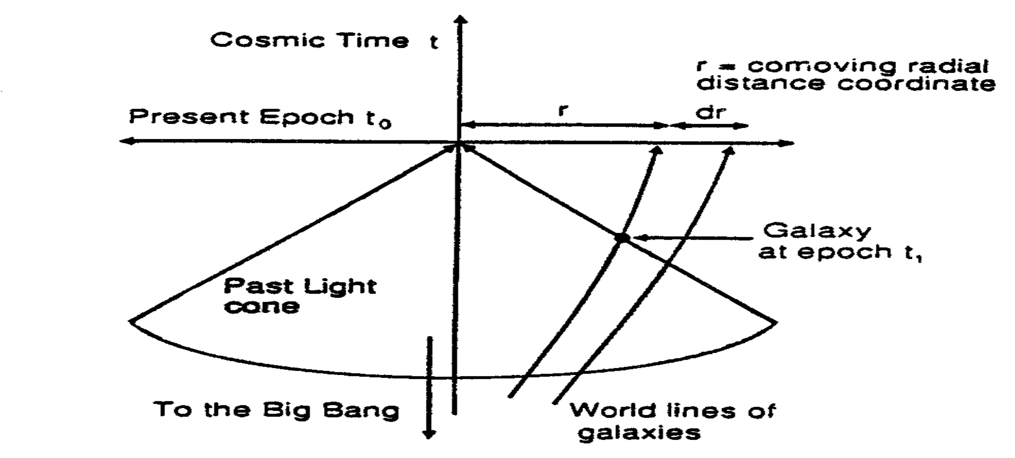
$$dl^2 = \frac{dx^2}{1-\kappa x^2} + x^2\left[d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right] \quad *$$

- Introduce time:
  - in Euclidean space (special relativity, Minkowski metric)

$$ds^2 = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right)$$

(c^2 = speed of light; dl^2 could be from *)

- We need to do this to understand observations of distant objects (distant = past) for photons: $ds^2 \equiv 0$!



- We have seen that the universe **cannot** be static, unless it is empty ⇒ time evolution.

If the universe is (and remains) homogeneous and isotropic ("cosmological principle"), then uniform expansion (or contraction) is the only alternative.

The radial distances to two points *i* and *j* vary with time as

$$\frac{\hat{\rho}_i(t_1)}{\hat{\rho}_j(t_1)} = \frac{\hat{\rho}_i(t_2)}{\hat{\rho}_j(t_2)} \Rightarrow \frac{\hat{\rho}_i(t_1)}{\hat{\rho}(t_2)_i} = \frac{a(t_1)}{a(t_2)}$$

for any observer and for any distance $\hat{\rho}_i$

$$\Rightarrow \quad \hat{\rho}(t) = a(t) \cdot r$$

r = "comoving" or "coordinate" distance,
a = expansion factor

<u>Transverse</u> separations

$$\frac{R_c(t)\sin\left(\hat{\rho}/R_c(t)\right)}{R_c(t_0)\sin\left(\hat{\rho}/R_c(t_0)\right)} = a(t)$$

with $t_0 = t_{(now)}$ and $a_{(now)} \equiv 1$

$$\Rightarrow Rc(t) = Rc(t_0) \cdot a(t) \equiv R \cdot a(t)$$

$$\boxed{\Rightarrow ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left[dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\cdot\left(d\vartheta^2 + \sin^2\vartheta\delta\varphi^2\right)\right]}$$

= Robertson-Walker Metric (RW-M)

R = present-day curvature
r = comoving radial coordinates
a(*t*) = expansion or scale factor
Subsumes all time dependence that is compatible with the cosmological principle.

Again, alternate forms are possible:

$$ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left[\frac{d\tilde{r}}{1-\kappa\tilde{r}^2} + \tilde{r}^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]$$

with $\tilde{r} = R\sin\left(\frac{r}{R}\right)$ "comoving angular diameter distance".

<u>Note</u>: There's always a transform $\kappa = \tilde{r}^2 \rightarrow \tilde{r}^2$ so that the curvature is *k* = -1, 0, 1.

- So far, the evolution of $a(t)$ is unspecified, i.e. no physics yet, just math.

- General relativity will determine $a(t)$ as a function of the mass (energy) density **and** link it to R!

- The "distances" r and $\tilde{r}$ are not observable, just coordinate distances.

II. 3 Observable Quantities in Cosmology

- In curved and expanding space:
  - app. size $\neq \frac{1}{\text{distance}}$
  - luminosity $\neq \frac{1}{\text{distance}^2}$
  - Is there a unique measure of distance, anyway?
- Some observables do not depend on the expansion history, $a(t)$, which we don't know (yet)!

II.3.1 Redshift

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

The right way of looking at it is **not**:

recession velocity $$(1+z) = \sqrt{\frac{1+v/c}{1-v/c}}$$

only for $v/c \ll 1$; $v = cz$

but, time dilation, or the universe's expansion during light travel:

- Consider two wave-crests, emitted at t_1 and $t_1 + \Delta\, t_1$.
- Light follows "null cone", i.e. $ds^2 = 0$ in RW-M, $(+\ d\vartheta = d\varphi = 0)$.

$$\Rightarrow \qquad dt = -\frac{a(t)}{c}\, dr$$

r = comoving coordinate

$$\Leftrightarrow \int_{t_1}^{t_0} \frac{c\,dt}{a(t)} = -\int_r^0 dr$$

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c\,dt}{a(t)} = -\int_r^0 dr$$

first crest (1)

second crest (2)

The light travel time difference is (b) – (a):

$$\frac{\Delta t_{obs}}{\Delta t_{emm}} = \frac{a\left(t_{obs}\right)}{a\left(t_{emm}\right)} = 1 + z$$

II.3.2 Hubble Constant

The Hubble constant, H_0 , is locally defined as $v_{recession} = H_0 \cdot d$, where d is the physical distance to the object.

$$\begin{aligned} v_{rec} &= z \cdot c = \left[\left(\frac{a\left(t_{obs}\right)}{a\left(t_{emm}\right)}\right) - 1\right] c \\ &= \dot{a}\left(t_{obs}\right) \cdot d \Rightarrow H_0 = \dot{a}\left(t_0\right) \end{aligned}$$

II. 4 General Relativity in (Not Even) a Nutshell

"One had to be a Newton to notice that the Moon is falling when everyone sees that it doesn't fall."

Paul Valéry

"In the spirit of the epigraph, we can say that it took the genius of Einstein to see that the Moon is moving in a straight line, when everyone sees that it doesn't."

W. Burke

1. Mass curves space

$$R_{ij} - \frac{R}{2} g_{ij} = 8\pi GT_{ij} \Lambda g_{ij}$$

Einstein's field equations

g_{ij} = metric tensor

R_{ij} = 1st and 2nd derivatives of g_{ij}

R = scalar curvature function

T_{ij} = energy-momentum tensor (energy including mc^2)in a locally Minkowski coordinate system (at rest w.r.t. a fluid)

Λ= "cosmological constant"

$$T_{ij} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

ρ = mass density

p = pressure

2. Space is locally "flat"

I.e. there is always a local coordinate transformation that transforms

$$g_{\mu\nu} \rightarrow \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

locally general relativity = special relativity

3. Particles follow geodesic world lines

- What is "geodesic"? It's the shortest path (world line) from a to b, i.e.

$\min\left(\int ds^2\right)$ (all possible paths)

- In flat space: $ds^2 = dt^2 - \frac{dx^2}{c^2}$

$\Rightarrow$ geodesics are straight lines (□ no force)

- In curved space
  - Each particle follows its geodesics.
  - Because $g_{\mu\nu} \neq$ constant, neighboring geodesics diverge (□ acceleration, force).

II.5 World Models and the „Friedman Equation“

Using Einstein's field equations and demanding homogeneity and isotropy, one finds:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3}$$

+energy conservation:

$d(\rho c^2 a^3) = -p d(a^3)$

<u>Note</u>:

ρ = total mass energy density (matter + photons!)

p = pressure

Continuity equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \quad \text{or } a\frac{d\rho}{da} = -3(\rho + P)$$

Using $H(t) = \frac{\dot{a}}{a}$ one finds

$$H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{K}{a^2}$$

K=integration constant

II.5.1 Classical Friedman Models

Let us first consider pressureless, or "dust", models p $\equiv$ 0 without cosmological constant, i.e. $\Lambda=0$:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho \cdot a^2 - \frac{c^2}{R^2}$$

(with $K=R^{-2}$)

Now identify the integration constant with the curvature. Define a critical density:

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} = 1.88 \cdot 10^{-29} \frac{g}{cm^3} h^2$$

Define $h = \frac{H_0}{100\ km/s/Mpc}$ and express the actual density in units of ρ_{crit}:

$$\Omega_0 \equiv \frac{8\pi G \rho_0}{3H_0^2} = \left(\frac{\rho_0}{\rho_c}\right)$$

$$\Rightarrow \quad \dot{a}^2 = \frac{\Omega_0 H_0^2}{a} - \frac{c^2}{R^2} \qquad \otimes$$

using that at present time a ≡ 1, $H_0 = \dot{a}$

and $\rho(t) = \rho_0 a^3(t)$

$$\Rightarrow \quad R = \frac{c / H_0}{(\Omega_0 - 1)^{1/2}}$$

⇒ Curvature $\kappa = \frac{1}{R^2}$ and mass density Ω_0 are linked!

In particular, space is flat if Ω_0 = 1 or $\rho = \rho_{crit}$

- spherical geometry: $\Leftrightarrow \Omega_0 > 1$, re-collapse
- hyperbolic geometry: $\Leftrightarrow \Omega_0 < 1$, infinite expansion
- flat space: $\Leftrightarrow \Omega_0 = 1$

The most convenient solution for $a(t)$ in $\oplus$ is in parametric form:

$a(\vartheta) = \alpha_1 \cdot (1 - \cos\vartheta)$ $t(\vartheta) = \alpha_2 \cdot (\vartheta - \sin\vartheta)$

$$\alpha_1 = \frac{\Omega_0}{2(\Omega_0 - 1)}$$

$$\alpha_2 = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}$$

How are redshift (observable) and time (not observable) related in these models?

Using
$$\dot{a} = \frac{8\pi G\rho_0}{3a} - \frac{c^2}{R^2}$$

and
$$a = \frac{1}{1+z}, \quad R = \frac{c/H_0}{(\Omega_0 - 1)^{1/2}}, \quad \Omega_0 = \rho / \left(3H_0^2/8\pi G\right)$$

we get
$$\frac{dz}{dt} = -H_0 (1+z)^2 (\Omega_0 z + 1)^{1/2}$$

So the time from the big bang to redshift *z* is

$$t = \int_0^t dt = -\frac{1}{H_0}\int_\infty^z \frac{dz}{(1+z)^2 (\Omega_0 z + 1)^{1/2}}$$

II.5.2 Models with a Cosmological Constant

Energy content of the universe:

a) $\rho_{mass} \sim a^{-3}$

b) $\rho_{radiation} \sim a^{-4}$

c) $\rho_{vac} = const. \Leftrightarrow \Omega_{vac} \equiv \Lambda c^2 / 3H_0^2$

$\Rightarrow$ In general

$$\frac{8\pi G\rho}{3} = H_0^2\left(\Omega_M a^{-3} + \Omega_{rad} a^{-4} + \Omega_{vac}\right)$$

or

$$H^2(a) = H_0^2\left[\Omega_{vac} + \Omega_M a^{-3} + \Omega_{rad} a^{-4} - (\Omega_0 - 1) a^{-2}\right]$$

where $(\Omega_0 - 1)$ reflects the curvature.

II.5.2 Simple Limiting Solutions

$$H^2(a) = H_0^2\left[\Omega_{vac} + \Omega_M a^{-3} + \Omega_{rad} a^{-4} - (\Omega_0 - 1)a^{-2}\right]$$

(1) $\left(\dfrac{\dot a}{a}\right)^2 = H_0^2\Omega_{rad} a^{-4} \leftrightarrow \dot a a = const \leftrightarrow a \sim t^{1/2}$ at early epochs

(2) $\left(\dfrac{\dot a}{a}\right)^2 = H_0^2\Omega_M a^{-3} \leftrightarrow a \sim t^{2/3}$ matter dominated

(3) $\left(\dfrac{\dot a}{a}\right)^2 = H_0^2\Omega_\Lambda \leftrightarrow a \sim e^t$ dominant Λ (late epochs)

II.6 General Solutions of the Friedman Equation

The Friedman equation can be written in the following form:

$$\frac{\dot{a}}{a} = H_0 \cdot E(z) = H_0 \cdot \sqrt{\Omega(1+z)^3 + \Omega_R(1+z) + \Omega_\Lambda}$$

with $\Omega \equiv \frac{8\pi G\rho_0}{3H_0}$, $\Omega_K = (H_0\, a_0\, R)^{-2}$, and $a=(1+z)^{-1}$

$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$ and $\Omega_{\mathrm{M+rad}} + \Omega_K + \Omega_\Lambda = 1.$

In flat space, the actual distance *D* equals the "coordinate distance" $\hat{\rho}$ with

$$\hat{\rho} = \frac{c}{H_0} \int_0^z \frac{dz}{E(z)}$$

In general,

$$D(z) = \frac{c}{H_0\sqrt{\Omega_R}} \sinh\left(\sqrt{\Omega_R}\,\frac{\hat{\rho} H_0}{c}\right) \quad \rightarrow \Omega_K > 0$$

$$D(z) = \hat{\rho}(z) \quad \rightarrow \Omega_K = 0$$

$$D(z) = \frac{c}{H_0\sqrt{-\Omega_R}} \sin\left(\sqrt{-\Omega_R}\,\frac{\hat{\rho} H_0}{c}\right) \quad \rightarrow \Omega_K < 0$$

Distance Measure(s) in Cosmology

present epoch Hubble constant	$H_0 = 100\,h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$
Hubble time	$t_\mathrm{H} \equiv \frac{1}{H_0} = 9.78 \times 10^9\,h^{-1}\ \mathrm{yr} = 3.09 \times 10^{17}\,h^{-1}\ \mathrm{s}$
Hubble radius/distance	$D_\mathrm{H} \equiv \frac{c}{H_0} = 3000\,h^{-1}\ \mathrm{Mpc} = 9.26 \times 10^{25}\,h^{-1}\ \mathrm{m}$
“Omega Matter”	$\Omega_\mathrm{M} \equiv \frac{8\pi\,G\,\rho_0}{3\,H_0^2}$
“Omega Lambda”	$\Omega_\Lambda \equiv \frac{\Lambda\,c^2}{3\,H_0^2}$
“equiv. Omega curvature”	$\Omega_\mathrm{M} + \Omega_\Lambda + \Omega_k = 1$
redshift	$z \equiv \frac{\nu_\mathrm{e}}{\nu_\mathrm{o}} - 1 = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} - 1$

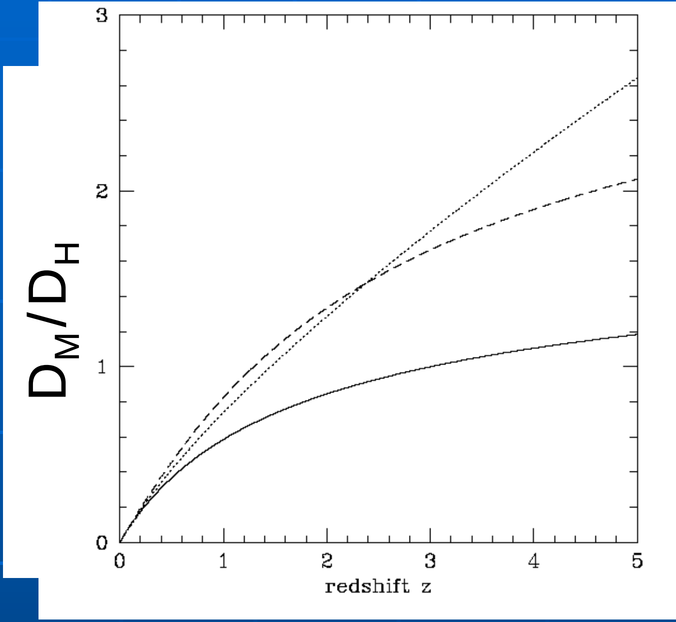
Comoving distance (line-of-sight)

=invariant under expansion

$$E(z) \equiv \sqrt{\Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda}$$

$$H(z) = H_0 E(z)$$

$$D_C = D_H \int_0^z \frac{dz'}{E(z')}$$



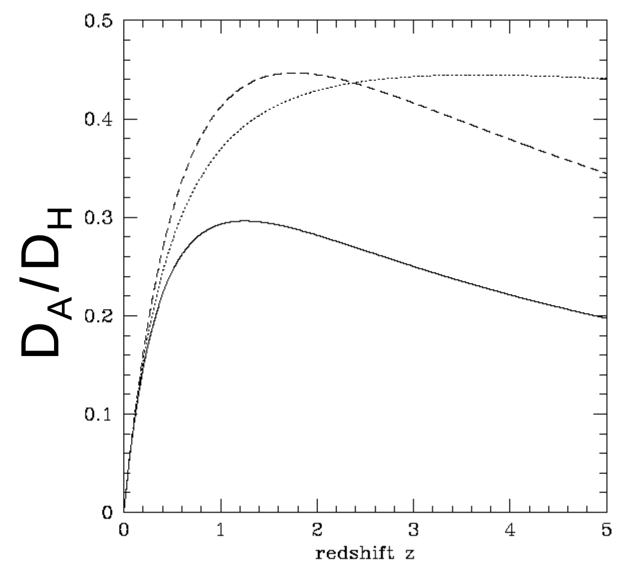
Comoving distance (transverse)

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left[\sqrt{\Omega_k}\, D_C/D_H\right] & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ D_H \frac{1}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|}\, D_C/D_H\right] & \text{for } \Omega_k < 0 \end{cases}$$

Angular diameter distance

$$D_A = \frac{D_M}{1+z}$$

= phys. size of object / observed angular size



Luminosity distance

The *luminosity distance* D_{L} is defined by the relationship between bolometric (ie, integrated over all frequencies) flux S and bolometric luminosity L:

$$D_{\mathrm{L}} \equiv \sqrt{\frac{L}{4\pi\, S}}$$

$$D_{\mathrm{L}} = (1+z)\, D_{\mathrm{M}} = (1+z)^2\, D_{\mathrm{A}}$$

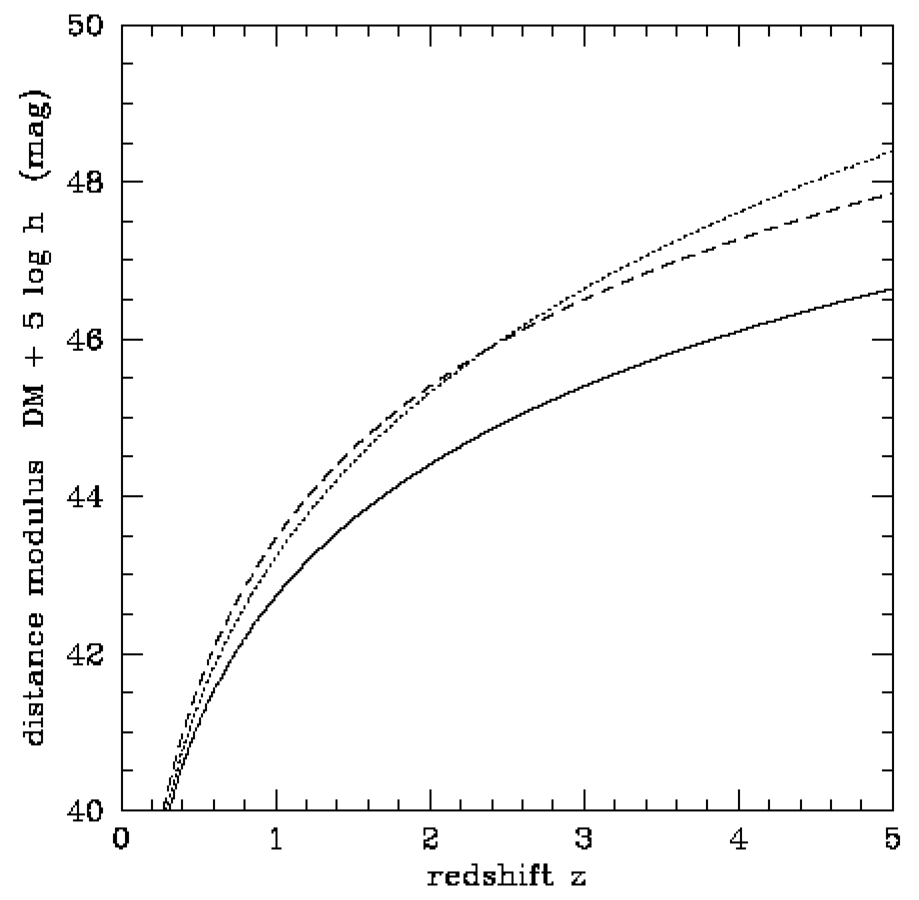
$$S_\nu = (1+z)\,\frac{L_{(1+z)\nu}}{L_\nu}\,\frac{L_\nu}{4\pi\, D_{\mathrm{L}}^2} \qquad S_\lambda = \frac{1}{(1+z)}\,\frac{L_{\lambda/(1+z)}}{L_\lambda}\,\frac{L_\lambda}{4\pi\, D_{\mathrm{L}}^2}$$

$$DM \equiv 5\,\log\left(\frac{D_{\mathrm{L}}}{10\ \mathrm{pc}}\right)$$

$$m = M + DM + K$$

K is the k-correction

$$K = -2.5\,\log\left[(1+z)\,\frac{L_{(1+z)\nu}}{L_\nu}\right] = -2.5\,\log\left[\frac{1}{(1+z)}\,\frac{L_{\lambda/(1+z)}}{L_\lambda}\right]$$

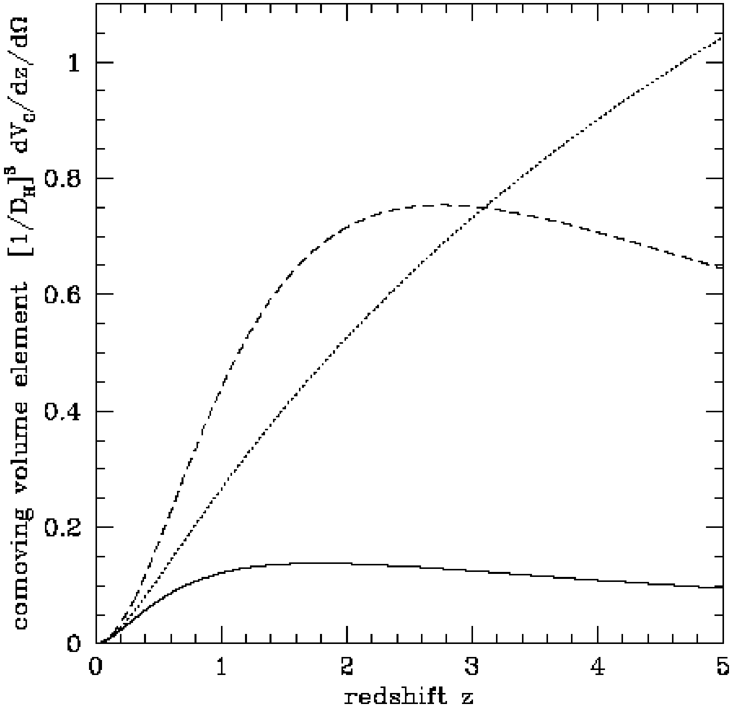

distance modulus DM + 5 log h (mag)
redshift z
40
42
44
46
48
50
0
1
2
3
4
5

Comoving volume

$$dV_{\rm C} = D_{\rm H} \frac{(1+z)^2\, D_{\rm A}^2}{E(z)}\, d\Omega\, dz$$

total comoving volume, all-sky, out to redshift z

$$V_{\rm C} = \begin{cases} \left(\frac{4\pi\, D_{\rm H}^3}{2\,\Omega_k}\right) \left[\frac{D_{\rm M}}{D_{\rm H}} \sqrt{1+\Omega_k\, \frac{D_{\rm M}^2}{D_{\rm H}^2}} - \frac{1}{\sqrt{|\Omega_k|}} \,\mathrm{arcsinh}\left(\sqrt{|\Omega_k|}\, \frac{D_{\rm M}}{D_{\rm H}}\right)\right] & \text{for } \Omega_k > 0 \\ \frac{4\pi}{3}\, D_{\rm M}^3 & \text{for } \Omega_k = 0 \\ \left(\frac{4\pi\, D_{\rm H}^3}{2\,\Omega_k}\right) \left[\frac{D_{\rm M}}{D_{\rm H}} \sqrt{1+\Omega_k\, \frac{D_{\rm M}^2}{D_{\rm H}^2}} - \frac{1}{\sqrt{|\Omega_k|}} \,\mathrm{arcsin}\left(\sqrt{|\Omega_k|}\, \frac{D_{\rm M}}{D_{\rm H}}\right)\right] & \text{for } \Omega_k < 0 \end{cases}$$



Lookback time

$$t_{\mathrm{L}} = t_{\mathrm{H}} \int_0^z \frac{dz'}{(1+z')\,E(z')}$$